

author: Imre Rábai

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given system of equations.

$$\begin{aligned}x^2 + 3xy &= -2 \\ 2xy - y^2 &= -5\end{aligned}$$

Solution: Multiplying the first equation by 5 and the second one by -2 , and adding them, the constants cancel:

$$5x^2 + 11xy + 2y^2 = 0, \quad (x + 2y)(5x + y) = 0$$

If $x = -2y$: the first equation gives $4y^2 - 6y^2 = -2$, so $y^2 = 1$: $(-2, 1)$ and $(2, -1)$.

If $y = -5x$: the first equation gives $x^2 - 15x^2 = -2$, so $x^2 = \frac{1}{7}$: $\left(\frac{1}{\sqrt{7}}, -\frac{5}{\sqrt{7}}\right)$ and $\left(-\frac{1}{\sqrt{7}}, \frac{5}{\sqrt{7}}\right)$.

All four pairs satisfy the second equation too, so the system has four solutions.

2. One root of the equation $x^2 + px + q = 0$ is 3, and the other root is equal to twice the discriminant of the equation. Find p and q .

Solution: Let $D = p^2 - 4q$ be the discriminant. The roots are 3 and $2D$, so by Viète's formulas $p = -(3 + 2D)$ and $q = 6D$. Then

$$D = (3 + 2D)^2 - 24D, \quad 4D^2 - 13D + 9 = 0, \quad D = 1 \text{ or } D = \frac{9}{4}$$

If $D = 1$: $p = -5$, $q = 6$, and the roots are 2 and 3. If $D = \frac{9}{4}$: $p = -\frac{15}{2}$, $q = \frac{27}{2}$, and the roots are $\frac{9}{2}$ and 3. Both are solutions.

3. Solve the given equation.

$$\log(4 - x) + 2 \log 3 = \log 108 - \frac{1}{2} \log x^2$$

Solution: We need $x < 4$ and $x \neq 0$. Since $\frac{1}{2} \log x^2 = \log |x|$, the equation is

$$\log(|x|(4 - x)) = \log 108 - \log 9 = \log 12, \quad |x|(4 - x) = 12$$

If $0 < x < 4$: $x^2 - 4x + 12 = 0$, which has no real solution. If $x < 0$: $-x(4 - x) = 12$, $x^2 - 4x - 12 = 0$, so $x = -2$ or $x = 6$, and only $x = -2$ is allowed. The solution is $x = -2$.

4. The radius of the circle circumscribed around triangle ABC is 5 units, the side AB is 8 units long, and the ratio of the other two sides is $2 : 5$. Find the other two sides of the triangle.

Solution: Let γ be the angle opposite AB . By the Law of Sines, $8 = 2 \cdot 5 \sin \gamma$, so $\sin \gamma = \frac{4}{5}$ and $\cos \gamma = \pm \frac{3}{5}$. Let the other two sides be $2x$ and $5x$. By the Law of Cosines, $64 = 4x^2 + 25x^2 - 20x^2 \cos \gamma$.

If $\cos \gamma = \frac{3}{5}$: $64 = 17x^2$, $x = \frac{8}{\sqrt{17}}$, and the sides are $\frac{16}{\sqrt{17}} \approx 3.88$ and $\frac{40}{\sqrt{17}} \approx 9.70$.

If $\cos \gamma = -\frac{3}{5}$: $64 = 41x^2$, $x = \frac{8}{\sqrt{41}}$, and the sides are $\frac{16}{\sqrt{41}} \approx 2.50$ and $\frac{40}{\sqrt{41}} \approx 6.25$.

Both triangles exist (for example, $3.88 + 8 > 9.70$ and $2.50 + 6.25 > 8$).

5. Find the equation of the circle that is tangent to the x -axis at the origin and is also tangent to the line $y = x + 2$.

Solution: The center is on the y -axis, $K(0, v)$, and the radius is $|v|$. The circle is tangent to the line $x - y + 2 = 0$ if the distance of K from it is the radius:

$$\frac{|0 - v + 2|}{\sqrt{2}} = |v|, \quad 2 - v = \pm \sqrt{2}v$$

So $v = \frac{2}{1 + \sqrt{2}} = 2\sqrt{2} - 2 \approx 0.83$ or $v = \frac{2}{1 - \sqrt{2}} = -2\sqrt{2} - 2 \approx -4.83$. There are two such circles:

$$x^2 + (y - 2\sqrt{2} + 2)^2 = (2\sqrt{2} - 2)^2, \quad x^2 + (y + 2\sqrt{2} + 2)^2 = (2\sqrt{2} + 2)^2$$

Note: The solution published in KoMaL solves the problem with the y -axis instead of the x -axis (and gets the circles $(x - 2 \mp 2\sqrt{2})^2 + y^2 = (2 \pm 2\sqrt{2})^2$).

6. Where does the function

$$f(x) = \sin^2 2x + 2 \cos^2 x - \frac{5}{4}$$

take its largest and its smallest values on the interval $[0, \pi]$? What are the largest and smallest values?

Solution: Using $\sin^2 2x = 1 - \cos^2 2x$ and $2 \cos^2 x = 1 + \cos 2x$,

$$f(x) = -\cos^2 2x + \cos 2x + \frac{3}{4} = 1 - \left(\cos 2x - \frac{1}{2}\right)^2$$

Since $-1 \leq \cos 2x \leq 1$, $-\frac{3}{2} \leq \cos 2x - \frac{1}{2} \leq \frac{1}{2}$, so $0 \leq \left(\cos 2x - \frac{1}{2}\right)^2 \leq \frac{9}{4}$, and $-\frac{5}{4} \leq f(x) \leq 1$.

The largest value 1 is taken where $\cos 2x = \frac{1}{2}$; in $[0, \pi]$ this is at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

The smallest value $-\frac{5}{4}$ is taken where $\cos 2x = -1$; in $[0, \pi]$ this is at $x = \frac{\pi}{2}$.

7. Depending on the value of the real number p , how many solutions does the equation

$$\sqrt{2|\log_2 x| - (\log_2 x)^2} = p$$

have?

Solution: Let $y = |\log_2 x| \geq 0$; then $(\log_2 x)^2 = y^2$, and the equation is $\sqrt{2y - y^2} = p$. Each value $y > 0$ gives two solutions $x = 2^y$ and $x = 2^{-y}$, and $y = 0$ gives one solution, $x = 1$.

If $p < 0$: no solution.

If $p \geq 0$: squaring, $y^2 - 2y + p^2 = 0$, so $y = 1 \pm \sqrt{1 - p^2}$ (if $p \leq 1$).

- $p = 0$: $y = 0$ or $y = 2$, so $x = 1, x = 4, x = \frac{1}{4}$: 3 solutions.
- $0 < p < 1$: two different positive values of y , so 4 solutions.
- $p = 1$: $y = 1$, so $x = 2$ and $x = \frac{1}{2}$: 2 solutions.
- $p > 1$: no solution.

8. The sides a, b , and c of a triangle satisfy the following relation.

$$\frac{1}{a+b} + \frac{1}{b-c} = \frac{3}{a+b-c}$$

Find the angle opposite the side b .

Solution: We need $b \neq c$ (and $a + b - c > 0$ by the triangle inequality). Multiplying by $(a+b)(b-c)(a+b-c)$:

$$(b-c)(a+b-c) + (a+b)(a+b-c) = 3(a+b)(b-c)$$

Expanding and simplifying, this becomes

$$a^2 + c^2 + ac = b^2$$

By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac \cos \beta$, so $-2ac \cos \beta = ac$, $\cos \beta = -\frac{1}{2}$, and the angle opposite the side b is $\beta = 120^\circ$.