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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The midline of a tangential trapezoid (a trapezoid with an inscribed circle) divides it into two parts whose areas are in the ratio 5 : 11. Find the lengths of the parallel sides of the trapezoid, if the legs are 3 and 5 units long.

Solution: A quadrilateral has an inscribed circle exactly when the sums of its opposite sides are equal, so the parallel sides add up to $3 + 5 = 8$, and the midline is 4 units long. Let the parallel sides be $4 - x$ and $4 + x$, and the height $2m$. The midline divides the trapezoid into two trapezoids of height m , so

$$\frac{\frac{(4-x)+4}{2}m}{\frac{(4+x)+4}{2}m} = \frac{8-x}{8+x} = \frac{5}{11}, \quad x = 3$$

The parallel sides are 1 and 7 units long. (Such a trapezoid exists: the legs project onto the longer base with lengths $\frac{5}{3}$ and $\frac{13}{3}$, and the height is $\frac{2\sqrt{14}}{3}$.)

2. Solve the following equations over the real numbers.

a) $2 \log_3 (x - 2) + 2 \log_3 (x - 4) = 0$

b) $2 \log_3 (x - 2) + \log_3 (x - 4)^2 = 0$

c) $\log_3 (x - 2)^2 + \log_3 (x - 4)^2 = 0$

Solution: a) We need $x > 4$. The equation says $(x - 2)(x - 4) = 1$, that is, $x^2 - 6x + 7 = 0$, so $x = 3 \pm \sqrt{2}$. Only $x = 3 + \sqrt{2}$ satisfies $x > 4$.

b) We need $x > 2$ and $x \neq 4$. The equation says $(x - 2)^2 (x - 4)^2 = 1$, that is, $(x - 2)|x - 4| = 1$. For $x > 4$ this gives $x = 3 + \sqrt{2}$ as in a); for $2 < x < 4$ it gives $(x - 2)(4 - x) = 1$, $(x - 3)^2 = 0$, $x = 3$. The solutions are $x = 3 + \sqrt{2}$ and $x = 3$.

c) We need $x \neq 2$ and $x \neq 4$. The equation says $|(x - 2)(x - 4)| = 1$, that is, $x^2 - 6x + 8 = \pm 1$. The solutions are $x = 3 + \sqrt{2}$, $x = 3 - \sqrt{2}$, and $x = 3$.

3. Find the equation of the circle whose tangent line at its point $A(3, 2)$ is $x + y = 5$, and whose center is 1 unit away from the origin.

Solution: The center is on the line through A perpendicular to the tangent, which is $x - y = 1$, so the center is $(3 + t, 2 + t)$ for some t . It is 1 unit from the origin:

$$(3 + t)^2 + (2 + t)^2 = 1, \quad t^2 + 5t + 6 = 0, \quad t = -2 \text{ or } t = -3$$

So the center is $C_1(1, 0)$ or $C_2(0, -1)$, and the radius squared is $AC_1^2 = 8$ or $AC_2^2 = 18$. There are two such circles:

$$(x - 1)^2 + y^2 = 8, \quad x^2 + (y + 1)^2 = 18$$

4. Find the value of m if the sum of the squares of the real roots of the equation

$$x^2 + (m - 2)x + m + 5 = 0$$

is 10.

Solution: By Viète's formulas, $x_1 + x_2 = 2 - m$ and $x_1x_2 = m + 5$, so

$$x_1^2 + x_2^2 = (2 - m)^2 - 2(m + 5) = m^2 - 6m - 6 = 10, \quad m^2 - 6m - 16 = 0$$

so $m = 8$ or $m = -2$. But the roots must be real: the discriminant is $(m - 2)^2 - 4(m + 5) = m^2 - 8m - 16$, which is $-16 < 0$ for $m = 8$ and $4 > 0$ for $m = -2$. So $m = -2$ (and the equation is $x^2 - 4x + 3 = 0$, with roots 1 and 3).

5. The product of the third and fifth terms of a geometric sequence is 9, and the sum of the second and sixth terms is $-\frac{51}{4}$. Find the first term and the ratio of the sequence.

Solution: Let the fourth term be a and the ratio $q \neq 0$. Then

$$\frac{a}{q} \cdot aq = a^2 = 9, \quad \frac{a}{q^2} + aq^2 = a \left(\frac{1}{q^2} + q^2 \right) = -\frac{51}{4}$$

So $a = \pm 3$; since $\frac{1}{q^2} + q^2 > 0$, we need $a < 0$, so $a = -3$, and $q^2 + \frac{1}{q^2} = \frac{17}{4}$, that is, $4q^4 - 17q^2 + 4 = 0$, so $q^2 = 4$ or $q^2 = \frac{1}{4}$. The first term is $a_1 = \frac{a}{q^3} = -\frac{3}{q^3}$. There are four such sequences:

$$a_1 = -\frac{3}{8}, q = 2; \quad a_1 = \frac{3}{8}, q = -2; \quad a_1 = -24, q = \frac{1}{2}; \quad a_1 = 24, q = -\frac{1}{2}$$

6. In trapezoid $ABCD$, one of the parallel sides is $BC = 4$ units, the legs are $AB = 2\sqrt{3}$ and $DC = \sqrt{10}$ units, and the angle BAD is 60° . Find the area of the trapezoid.

Solution: Let B_1 and C_1 be the projections of B and C onto the line AD . In the right triangle ABB_1 , the angle at A is 60° , so $AB_1 = \sqrt{3}$ and the height is $BB_1 = 3$. In the right triangle CC_1D , $C_1D^2 = 10 - 9 = 1$, so $C_1D = 1$, and D can be on either side of C_1 :

$$AD = \sqrt{3} + 4 + 1 = 5 + \sqrt{3} \quad \text{or} \quad AD = \sqrt{3} + 4 - 1 = 3 + \sqrt{3}$$

Both give a trapezoid, and the areas are

$$\frac{4 + 5 + \sqrt{3}}{2} \cdot 3 = \frac{3(9 + \sqrt{3})}{2} \approx 16.10 \quad \text{or} \quad \frac{4 + 3 + \sqrt{3}}{2} \cdot 3 = \frac{3(7 + \sqrt{3})}{2} \approx 13.10$$

7. The areas of the base and the top of a frustum of a pyramid are t and T . We cut the frustum with the plane that is halfway between the base and the top (parallel to them). Express the area of the cross-section in terms of t and T .

Solution: Complete the frustum to a pyramid. Cross-sections of a pyramid parallel to the base are similar to the base, and their linear sizes are proportional to their distances from the apex; so the square roots of their areas are proportional to these distances. The cross-section halfway between the base and the top is at the average of the distances of the base and the top from the apex, so with τ the area of the cross-section,

$$\sqrt{\tau} = \frac{\sqrt{t} + \sqrt{T}}{2}, \quad \tau = \left(\frac{\sqrt{t} + \sqrt{T}}{2} \right)^2 = \frac{t + 2\sqrt{tT} + T}{4}$$

8. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned} \cos y &= \sqrt{2} \cos x \\ \sin y &= \sqrt{\frac{2}{3}} \sin x \end{aligned}$$

Solution: Squaring both equations and adding them:

$$1 = 2 \cos^2 x + \frac{2}{3} \sin^2 x = \frac{2}{3} + \frac{4}{3} \cos^2 x, \quad \cos^2 x = \frac{1}{4}$$

Then $\sin^2 x = \frac{3}{4}$, so $\cos y = \sqrt{2} \cos x = \pm \frac{\sqrt{2}}{2}$ and $\sin y = \sqrt{\frac{2}{3}} \sin x = \pm \frac{\sqrt{2}}{2}$, where $\cos y$ has the same sign as $\cos x$, and $\sin y$ has the same sign as $\sin x$. So y is in the same quadrant as x . The solutions are (k and n are any integers):

$$\begin{aligned} x &= \frac{\pi}{3} + 2k\pi, & y &= \frac{\pi}{4} + 2n\pi \\ x &= \frac{2\pi}{3} + 2k\pi, & y &= \frac{3\pi}{4} + 2n\pi \\ x &= \frac{4\pi}{3} + 2k\pi, & y &= \frac{5\pi}{4} + 2n\pi \\ x &= \frac{5\pi}{3} + 2k\pi, & y &= \frac{7\pi}{4} + 2n\pi \end{aligned}$$