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Pdf file: KoMaL, October 1990, page 295

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the exact values of the following expressions.

$$\text{a) } \frac{\log_3 90}{\log_{30} 3} - \frac{\log_3 270}{\log 3} \qquad \text{b) } \frac{\left(1 - \tan^2 \frac{\pi}{8}\right) \left(2 \sin^2 \frac{5\pi}{12} - 1\right)}{2 \tan \frac{\pi}{8}}$$

Solution: a) Let $L = \log_3 90$ and $T = \log_3 10$. Then $\frac{1}{\log_{30} 3} = \log_3 30 = 1 + T$, $\frac{1}{\log 3} = \log_3 10 = T$, and $\log_3 270 = 1 + L$. So the expression is

$$L(1 + T) - (1 + L)T = L - T = \log_3 \frac{90}{10} = \log_3 9 = 2$$

b) By the double angle formulas,

$$\frac{1 - \tan^2 \frac{\pi}{8}}{2 \tan \frac{\pi}{8}} = \frac{1}{\tan \frac{\pi}{4}} = 1, \quad 2 \sin^2 \frac{5\pi}{12} - 1 = -\cos \frac{5\pi}{6} = \frac{\sqrt{3}}{2}$$

so the value of the expression is $\frac{\sqrt{3}}{2}$.

2. The product of the second and sixth terms of a geometric sequence is 576, and the difference of the fifth and third terms is 36. Write down the first six terms of the sequence.

Solution: Let the fourth term be a and the ratio $q \neq 0$. Then $a_2 a_6 = \frac{a}{q^2} \cdot a q^2 = a^2 = 576$, so $a = \pm 24$, and

$$a_5 - a_3 = a \left(q - \frac{1}{q} \right) = 36.$$

If $a = 24$: $q - \frac{1}{q} = \frac{3}{2}$, $2q^2 - 3q - 2 = 0$, so $q = 2$ or $q = -\frac{1}{2}$.

If $a = -24$: $q - \frac{1}{q} = -\frac{3}{2}$, $2q^2 + 3q - 2 = 0$, so $q = -2$ or $q = \frac{1}{2}$.

There are four such sequences:

$$\begin{aligned} q = 2 : & \quad 3, 6, 12, 24, 48, 96 \\ q = -\frac{1}{2} : & \quad -192, 96, -48, 24, -12, 6 \\ q = -2 : & \quad 3, -6, 12, -24, 48, -96 \\ q = \frac{1}{2} : & \quad -192, -96, -48, -24, -12, -6 \end{aligned}$$

3. Find the equation of the line through the point $P(2, 2)$ that is tangent to the circle $x^2 + y^2 = 2x$.

Solution: The circle is $(x - 1)^2 + y^2 = 1$, with center $K(1, 0)$ and radius 1. The point P is outside the circle, so there are two tangent lines through it. The vertical line $x = 2$ is 1 unit away from K , so it is a tangent. Any other line through P is $y - 2 = m(x - 2)$, that is, $mx - y + 2 - 2m = 0$, and it is tangent if its distance from K is 1:

$$\frac{|m + 2 - 2m|}{\sqrt{m^2 + 1}} = 1, \quad (2 - m)^2 = m^2 + 1, \quad m = \frac{3}{4}$$

The two tangent lines are

$$x = 2 \quad \text{and} \quad y - 2 = \frac{3}{4}(x - 2), \text{ that is, } 3x - 4y + 2 = 0$$

4. The area of a triangle is 40 square units, and two of its medians are 5 and 12 units long. How long is the third median?

Solution: Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB , and S the centroid, with $AA_1 = 12$ and $BB_1 = 5$. The triangle ASB has one third of the area of ABC (its height from S is one third of the height from C), and $AS = \frac{2}{3} \cdot 12 = 8$, $BS = \frac{2}{3} \cdot 5 = \frac{10}{3}$. With $\varphi = \angle ASB$:

$$\frac{40}{3} = \frac{1}{2} \cdot 8 \cdot \frac{10}{3} \sin \varphi, \quad \sin \varphi = 1$$

So $\varphi = 90^\circ$, and $AB = \sqrt{64 + \frac{100}{9}} = \frac{26}{3}$. In the right triangle ASB , the median SC_1 to the hypotenuse is half of the hypotenuse: $SC_1 = \frac{13}{3}$. The centroid divides the median CC_1 in the ratio 2 : 1, so

$$CC_1 = 3 \cdot SC_1 = 13$$

5. The sides of a triangle are a, b, c , and the opposite angles are α, β, γ . Find the exact value of $\sin \beta$ if $a + c = 2b$ and $\gamma = \alpha + 90^\circ$.

Solution: By the Law of Sines, $a + c = 2b$ means $\sin \alpha + \sin \gamma = 2 \sin \beta$. Using the sum-to-product formula and the double angle formula:

$$2 \sin \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} = 4 \sin \frac{\beta}{2} \cos \frac{\beta}{2}$$

Here $\frac{\alpha + \gamma}{2} = 90^\circ - \frac{\beta}{2}$, so $\sin \frac{\alpha + \gamma}{2} = \cos \frac{\beta}{2}$, and $\frac{\alpha - \gamma}{2} = -45^\circ$, so $\cos \frac{\alpha - \gamma}{2} = \frac{\sqrt{2}}{2}$. Dividing by $2 \cos \frac{\beta}{2} > 0$:

$$\frac{\sqrt{2}}{2} = 2 \sin \frac{\beta}{2}, \quad \sin \frac{\beta}{2} = \frac{\sqrt{2}}{4}, \quad \cos \frac{\beta}{2} = \sqrt{1 - \frac{2}{16}} = \frac{\sqrt{14}}{4}$$

So

$$\sin \beta = 2 \cdot \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{14}}{4} = \frac{\sqrt{7}}{4}$$

6. The segment AA' is a diameter of the circle circumscribed around triangle ABC . Reflect A' through the midpoint A_1 of the side BC . Prove that this reflection is the orthocenter of the triangle.

Solution: Let M be the reflection of A' through A_1 . The diagonals of the quadrilateral $A'BMC$ bisect each other at A_1 , so it is a parallelogram: $BM \parallel A'C$ and $CM \parallel A'B$. Since AA' is a diameter, by Thales' theorem $A'C \perp AC$ and $A'B \perp AB$. So $BM \perp AC$ and $CM \perp AB$: the lines BM and CM are altitudes of the triangle, and their intersection M is the orthocenter.

(If A' coincides with B or C , that is, if the triangle has a right angle at C or B , then M is that vertex, which is again the orthocenter.)

7. Solve the given equation over the real numbers.

$$2^{2x+1} = 3 \cdot 2^x + \sqrt{4 - 2^{x+3} + 2^{2(x+1)}}$$

Solution: Let $u = 2^{x+1} > 0$. Then $2^{x+3} = 4u$ and $2^{2(x+1)} = u^2$, so the expression under the square root is $u^2 - 4u + 4 = (u - 2)^2$, and its square root is $|u - 2|$. Also $2^{2x+1} = \frac{u^2}{2}$ and $3 \cdot 2^x = \frac{3u}{2}$. Multiplying by 2:

$$u^2 - 3u = 2|u - 2|$$

If $u \geq 2$: $u^2 - 5u + 4 = 0$, so $u = 4$ (the root $u = 1$ is less than 2).

If $u < 2$: $u^2 - u - 4 = 0$, so $u = \frac{1 \pm \sqrt{17}}{2}$, but $\frac{1 + \sqrt{17}}{2} > 2$ and $\frac{1 - \sqrt{17}}{2} < 0$, so there is no solution here.

So $2^{x+1} = 4$, and the only solution is $x = 1$. (Check: $8 = 6 + \sqrt{4 - 16 + 16}$.)

8. For what values of the real parameter m does the equation

$$\frac{2m \cdot \cos 2x - 4(m - 1) \cdot \cos x + 3m - 2}{1 - 2 \cdot \cos x} = 0$$

have a real solution?

Solution: We need the numerator to be 0 and $\cos x \neq \frac{1}{2}$. With $c = \cos x$ and $\cos 2x = 2c^2 - 1$, the numerator is

$$4mc^2 - 4(m - 1)c + m - 2$$

One can check that $c = \frac{1}{2}$ always makes it zero. If $m = 0$, the numerator is $4c - 2$, which is zero only for $c = \frac{1}{2}$, so there is no solution. If $m \neq 0$, the product of the roots is $\frac{m - 2}{4m}$, so the other root is

$$c = \frac{m - 2}{2m} = \frac{1}{2} - \frac{1}{m}$$

which is never $\frac{1}{2}$. So the equation has a solution if and only if $-1 \leq \frac{1}{2} - \frac{1}{m} \leq 1$, that is, $-\frac{1}{2} \leq \frac{1}{m} \leq \frac{3}{2}$, which gives

$$m \leq -2 \quad \text{or} \quad m \geq \frac{2}{3}$$