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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

There are 180 minutes to solve and present all problems.

1. Which one is greater?

$$\text{a) } \sqrt{4 - 2\sqrt{3}} - \sqrt{4 + 2\sqrt{3}} \quad \text{or} \quad \frac{\tan \frac{2\pi}{3}}{\sin \left(-\frac{5\pi}{3} \right)}$$

$$\text{b) } 2 \log_2 5 \quad \text{or} \quad 3 \left(1 + \log_{\frac{1}{8}} \frac{1}{3} \right)$$

Solution: a) $4 \mp 2\sqrt{3} = (\sqrt{3} \mp 1)^2$, so the first number is $(\sqrt{3} - 1) - (\sqrt{3} + 1) = -2$. Also $\tan \frac{2\pi}{3} = -\sqrt{3}$ and $\sin \left(-\frac{5\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so the second number is $\frac{-\sqrt{3}}{\sqrt{3}/2} = -2$. The two numbers are equal.

b) $\log_{\frac{1}{8}} \frac{1}{3} = \log_8 3 = \frac{1}{3} \log_2 3$, so the second number is $3 + \log_2 3 = \log_2 24$. The first number is $\log_2 25$, so the first number, $2 \log_2 5$, is greater.

Note: In KoMaL, the base of the logarithm in b) is printed as 1,8 (that is, 1.8), which is most likely a misprint for $\frac{1}{8}$; the published solution uses $\frac{1}{8}$.

2. Prove that if the angles α, β, γ of a triangle satisfy

$$\frac{\sin^2 \alpha}{\sin 2\alpha} = \frac{\sin^2 \beta}{\sin 2\beta} = \frac{\sin^2 \gamma}{\sin 2\gamma}$$

then the triangle is equilateral.

Solution: The fractions are defined, so no angle is 90° . Since $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ and $\sin \alpha \neq 0$, each fraction is $\frac{\sin \alpha}{2 \cos \alpha} = \frac{\tan \alpha}{2}$. So $\tan \alpha = \tan \beta = \tan \gamma$. The tangent function takes each value only once on $(0^\circ, 180^\circ)$ (apart from 90° , where it is not defined), so $\alpha = \beta = \gamma = 60^\circ$, and the triangle is equilateral.

3. Solve the following equations over the real numbers.

a) $\frac{x-3}{\sqrt{x+1}-2} = \sqrt{x+1} + 2$

b) $\log(4-x^2) - \log(2+x) = \log(2-x)$

c) $\frac{\sin \alpha - \frac{1}{2} \sin(\pi + 2\alpha)}{1 + \cos \alpha} = \sin \alpha$

Solution: All three equations are true for every allowed value of the variable.

a) We need $x \geq -1$ and $\sqrt{x+1} \neq 2$, that is, $x \neq 3$. For these x ,

$$\frac{x-3}{\sqrt{x+1}-2} = \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4} = \sqrt{x+1} + 2$$

so the solution is every $x \geq -1$ with $x \neq 3$.

b) We need $-2 < x < 2$. For these x , $\log(4-x^2) = \log(2-x) + \log(2+x)$, so the solution is every x with $-2 < x < 2$.

c) We need $\cos \alpha \neq -1$, that is, $\alpha \neq \pi + 2k\pi$. Since $\sin(\pi + 2\alpha) = -\sin 2\alpha = -2\sin \alpha \cos \alpha$, the numerator is $\sin \alpha (1 + \cos \alpha)$, so the solution is every $\alpha \neq \pi + 2k\pi$.

Note: In KoMaL, the numerator in a) is printed as $\sqrt{x-3}$. The published solution uses $x-3$, which fits the pattern of the other two parts; with $\sqrt{x-3}$, the equation becomes $\sqrt{x-3} = x-3$, and its only solution is $x = 4$.

4. Solve the system of equations

$$\begin{aligned} x - y &= a + 1 \\ 2xy + 2y &= -a^2 - 4 \end{aligned}$$

where a is a real parameter.

Solution: From the first equation $x = y + a + 1$; substituting into the second one:

$$2y(y + a + 1) + 2y + a^2 + 4 = 0, \quad 2y^2 + 2(a+2)y + a^2 + 4 = 0$$

The discriminant is $4(a+2)^2 - 8(a^2 + 4) = -4(a-2)^2$. So if $a \neq 2$, there is no solution; if $a = 2$, then $y = -2$, $x = 1$ is the only solution.

5. The perimeter of a right triangle is 15 units, and the radius of its inscribed circle is 1 unit. Find the sides of the triangle.

Solution: Let the legs be a and b and the hypotenuse c . The inscribed circle touches the legs 1 unit from the right-angle vertex, so the other tangent segments are $a-1$ and $b-1$, and by the equality of tangent segments, $c = (a-1) + (b-1)$.

With $a + b + c = 15$, this gives $c = \frac{13}{2}$ and $a + b = \frac{17}{2}$. By the Pythagorean theorem,

$$2ab = (a+b)^2 - c^2 = \frac{289 - 169}{4} = 30, \quad ab = 15$$

So a and b are the roots of $t^2 - \frac{17}{2}t + 15 = 0$, that is, 6 and $\frac{5}{2}$. The sides are $\frac{5}{2}$, 6, and $\frac{13}{2}$ units.

6. In rectangle $ABCD$, $3 \cdot AB = 2 \cdot BC$. Find the coordinates of the points C and D , if $A(-2, -2)$ and $B(4, 2)$.

Solution: $\overrightarrow{AB} = (6, 4)$, and $\overrightarrow{BC} = \overrightarrow{AD}$ is perpendicular to it, with $BC = \frac{3}{2}AB$. Rotating $(6, 4)$ by $\pm 90^\circ$ gives $\pm(-4, 6)$, so $\overrightarrow{BC} = \pm(-6, 9)$. There are two such rectangles:

$$\overrightarrow{BC} = (-6, 9) : C(-2, 11), D(-8, 7)$$

$$\overrightarrow{BC} = (6, -9) : C(10, -7), D(4, -11)$$

7. In the arithmetic sequence (a_n) , $a_3 = 3a_2$ and $s_n = 40a_3$, where s_n is the sum of the first n terms. Find the n th term of the sequence.

Solution: Let the difference be d . Then $a_2 + d = 3a_2$, so $a_2 = \frac{d}{2}$, $a_1 = -\frac{d}{2}$, and $a_3 = \frac{3d}{2}$. The second condition:

$$\frac{n}{2}(-d + (n-1)d) = 40 \cdot \frac{3d}{2}, \quad d(n^2 - 2n - 120) = 0$$

If $d = 0$, every term is 0. Otherwise $n^2 - 2n - 120 = (n-12)(n+10) = 0$, so $n = 12$, and

$$a_{12} = -\frac{d}{2} + 11d = \frac{21}{2}d$$

(that is, $a_{12} = 7a_3$).

8. Solve the following inequalities over the real numbers.

a) $2^{\sqrt{x+2}} < 3^{\sqrt{5} \cdot \log_3 2}$

b) $\log_{1/2}(x^2 - 3x - 2) < -1$

c) $\sin(\pi - x) \cdot \sin\left(\frac{3\pi}{2} + x\right) > \frac{\sqrt{3}}{4}$

Solution: a) $3^{\sqrt{5} \log_3 2} = (3^{\log_3 2})^{\sqrt{5}} = 2^{\sqrt{5}}$, so the inequality is $\sqrt{x+2} < \sqrt{5}$, that is, $0 \leq x+2 < 5$: $-2 \leq x < 3$.

b) Since $-1 = \log_{1/2} 2$ and $\log_{1/2}$ is decreasing, we need $x^2 - 3x - 2 > 2$, that is, $(x+1)(x-4) > 0$: $x < -1$ or $x > 4$.

c) $\sin(\pi - x) = \sin x$ and $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$, so the left-hand side is $-\sin x \cos x = -\frac{1}{2} \sin 2x$, and the inequality is $\sin 2x < -\frac{\sqrt{3}}{2}$. This means $\frac{4\pi}{3} + 2k\pi < 2x < \frac{5\pi}{3} + 2k\pi$, so

$$\frac{2\pi}{3} + k\pi < x < \frac{5\pi}{6} + k\pi, \quad k \text{ is an integer}$$