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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations.

$$\begin{aligned}\sin x &= 2 \sin y \\ x + y &= \frac{2\pi}{3}\end{aligned}$$

Solution: From the second equation $y = \frac{2\pi}{3} - x$. Substituting into the first one:

$$\sin x = 2 \sin \left(\frac{2\pi}{3} - x \right) = 2 \left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x \right) = \sqrt{3} \cos x + \sin x$$

so $\cos x = 0$. The solutions are

$$x = \frac{\pi}{2} + n\pi, \quad y = \frac{\pi}{6} - n\pi, \quad n \text{ is an integer}$$

(all steps were equivalent, so these are indeed solutions).

2. Solve the following inequalities.

a) $\sqrt{x^2 - 5x + 4} > -1$

b) $\log_{\frac{1}{2}}(4^x - 5 \cdot 2^x + 8) < -2$

Solution: a) A square root is never negative, so the inequality is true wherever the square root is defined: $x^2 - 5x + 4 = (x - 1)(x - 4) \geq 0$. The solution is $x \leq 1$ or $x \geq 4$.

b) Since $-2 = \log_{\frac{1}{2}} 4$ and the function $\log_{\frac{1}{2}}$ is decreasing, the inequality means $4^x - 5 \cdot 2^x + 8 > 4$ (and then the argument is automatically positive). With $t = 2^x > 0$:

$$t^2 - 5t + 4 > 0, \quad (t - 1)(t - 4) > 0, \quad t < 1 \text{ or } t > 4$$

So the solution is $x < 0$ or $x > 2$.

3. In triangle ABC , $AB = \sqrt{3}$, $AC = \sqrt{2}$ units, and $\angle BAC = 75^\circ$. On the arc BC of the circumscribed circle that does not contain A , mark the point D so that $\angle BAD = 30^\circ$. Find the length of the segment AD .

Solution: Let r be the radius of the circumscribed circle and $x = AD$. Since $\angle BAD = 30^\circ$, $\angle DAC = 45^\circ$, and by the Law of Sines (in the triangles ABD and ADC , which have the same circumscribed circle)

$$BD = 2r \sin 30^\circ = r, \quad CD = 2r \sin 45^\circ = r\sqrt{2}$$

By the Law of Cosines in the triangles ABD and ADC :

$$r^2 = 3 + x^2 - 2\sqrt{3}x \cos 30^\circ = 3 + x^2 - 3x, \quad 2r^2 = 2 + x^2 - 2\sqrt{2}x \cos 45^\circ = 2 + x^2 - 2x$$

Subtracting twice the first equation from the second one: $0 = -4 - x^2 + 4x$, that is, $(x - 2)^2 = 0$. So $AD = 2$ units (and $r = 1$).

(Another way: $ABDC$ is a cyclic quadrilateral, so by Ptolemy's theorem $AD \cdot BC = AB \cdot CD + AC \cdot BD$.)

4. The difference of an arithmetic sequence is $\frac{1}{2}$. The sum of the first n terms is 38, and the sum of the first $n + 4$ terms is 69. Find the value of n and the first term of the sequence.

Solution: Using $S_k = \frac{k}{2} (2a_1 + (k - 1)d)$ with $d = \frac{1}{2}$:

$$38 = \frac{n}{2} \left(2a_1 + \frac{n-1}{2} \right), \quad 69 = \frac{n+4}{2} \left(2a_1 + \frac{n+3}{2} \right)$$

From the first equation $2a_1 = \frac{76}{n} - \frac{n-1}{2}$; substituting into the second one,

$$138 = (n+4) \left(\frac{76}{n} + 2 \right), \quad n^2 - 27n + 152 = 0, \quad (n-8)(n-19) = 0$$

If $n = 8$: $a_1 = 3$. If $n = 19$: $a_1 = -\frac{5}{2}$. Both work.

5. The volume of the sphere inscribed in a right circular cone is one third of the volume of the cone. What fraction of the surface area of the cone is the surface area of the sphere?

Solution: The answer is $\frac{1}{3}$.

Let r be the radius of the sphere and O its center. Cut the cone into thin pieces from O : each piece is like a pyramid (or cone) with apex O , whose base is a small part of the surface of the cone, and whose height is r (the surface of the cone is tangent to the sphere everywhere). So the volume of the cone is $V = \frac{1}{3}rA$, where A is its surface area. For the sphere, similarly, $V_{\text{sphere}} = \frac{4}{3}r^3\pi = \frac{1}{3}r \cdot 4r^2\pi$. So

$$\frac{A_{\text{sphere}}}{A} = \frac{V_{\text{sphere}}}{V} = \frac{1}{3}$$

(With a computation: let R be the radius of the base, m the height, and a the slant height. The volume condition is $12r^3 = R^2m$. In the cross-section through the axis, the area of the triangle is $Rm = r(a + R)$ (half the perimeter times the inscribed radius). So $R(R + a) = \frac{R^2m}{r} = 12r^2$, and the ratio of the surface areas is $\frac{4r^2\pi}{R\pi(R + a)} = \frac{4r^2}{12r^2} = \frac{1}{3}$.)

6. A parabola with axis parallel to the y -axis passes through the point $A(4, -7)$ and is tangent to the line $y = 1$. A normal vector of the tangent line to the parabola at A is $\mathbf{n}(8, 1)$. Find the equation of the parabola.

Solution: A horizontal line tangent to a parabola with vertical axis goes through the vertex, so the parabola is $y = a(x - u)^2 + 1$. The tangent at A has normal vector $(8, 1)$, so its slope is -8 . The slope of the tangent at $x = 4$ is $2a(4 - u)$ (this can be found with derivatives, or from the discriminant of the intersection of the parabola and the line $8x + y = 25$). So

$$a(4 - u)^2 = -8, \quad 2a(4 - u) = -8$$

Dividing: $4 - u = 2$, so $u = 2$ and $a = -2$. The parabola is

$$y = -2(x - 2)^2 + 1$$

7. Solve the equation

$$\sqrt{x^2 + 3x + 2 + \frac{p^2}{x^2}} = x + 1 - \frac{p}{x}$$

where p is a real parameter.

Solution: We need $x \neq 0$ and $x + 1 - \frac{p}{x} \geq 0$; under this condition, squaring is an equivalent step:

$$x^2 + 3x + 2 + \frac{p^2}{x^2} = x^2 + 2x + 1 - 2(x + 1)\frac{p}{x} + \frac{p^2}{x^2}, \quad (x + 1)\left(1 + \frac{2p}{x}\right) = 0$$

So $x = -1$ or $x = -2p$.

$x = -1$: the right-hand side is p , so this is a solution if $p \geq 0$.

$x = -2p$ (with $p \neq 0$): the right-hand side is $-2p + 1 + \frac{1}{2} = \frac{3}{2} - 2p$, so this is a solution if $p \leq \frac{3}{4}$.

Summary:

- $p < 0$: $x = -2p$.
- $p = 0$: $x = -1$.
- $0 < p \leq \frac{3}{4}$: $x = -1$ and $x = -2p$ (these are the same for $p = \frac{1}{2}$).
- $p > \frac{3}{4}$: $x = -1$.

8. The perimeter of triangle ABC is $2s = 24$ units. We draw the tangent lines to its inscribed circle that are parallel to the sides. Of the segments of these tangent lines inside the triangle, choose the longest. For which triangles ABC is this tangent segment as long as possible?

Solution: Let $a = BC$, and let the tangent parallel to BC cut off the small triangle $AB'C'$ with $B'C' = x$. The two tangent segments from A to the inscribed circle have length $s - a$, and the tangent segments from B' and C' add up to x , so the perimeter of the small triangle is $2(s - a) = 2(12 - a)$. The small triangle is similar to ABC , so

$$\frac{x}{a} = \frac{2(12 - a)}{24}, \quad x = \frac{a(12 - a)}{12} = \frac{36 - (a - 6)^2}{12} \leq 3$$

with equality for $a = 6$. The same holds for the other two tangent segments. So each tangent segment is at most 3, and the longest one is 3 exactly when one side of the triangle is 6 units long (one fourth of the perimeter). Such triangles exist: $a = 6$ and $b + c = 18$, for example $b = c = 9$.