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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In the regular hexagon $ABCDEF$, let G be the midpoint of the side BC , and H the midpoint of the diagonal DF . Prove that the triangle AGH is equilateral. What is the ratio of the areas of the triangle and the hexagon?

Solution: Let the side of the hexagon be 1, with center at the origin and $A(1, 0)$, $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $D(-1, 0)$, $E\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $F\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Then $G\left(0, \frac{\sqrt{3}}{2}\right)$ and $H\left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$, and

$$AG^2 = 1 + \frac{3}{4} = \frac{7}{4}, \quad AH^2 = \frac{25}{16} + \frac{3}{16} = \frac{7}{4}, \quad GH^2 = \frac{1}{16} + \frac{27}{16} = \frac{7}{4}$$

so the triangle AGH is equilateral. (Without coordinates: the rotation by 60° about A takes G to H .) Its area is $\frac{\sqrt{3}}{4} \cdot \frac{7}{4} = \frac{7\sqrt{3}}{16}$, and the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{24\sqrt{3}}{16}$, so the ratio of the areas is $7 : 24$.

2. For the four-digit number $\overline{abba}$, we know that $27(\overline{ba} - \overline{ab}) + 47 = \overline{abba}$. What is this four-digit number?

Solution: $\overline{ba} - \overline{ab} = (10b + a) - (10a + b) = 9(b - a)$ and $\overline{abba} = 1001a + 110b$, so

$$243(b - a) + 47 = 1001a + 110b, \quad 1244a = 133b + 47$$

Since $b \leq 9$, the right-hand side is at most 1244, and $a \geq 1$, so $a = 1$ and $133b = 1197$, $b = 9$. The number is 1991. (Check: $27(91 - 19) + 47 = 1944 + 47 = 1991$.)

3. For which x is the expression $\sqrt{\log(x^2 + 2x - 15)}$ defined?

Solution: We need $\log(x^2 + 2x - 15) \geq 0$, that is, $x^2 + 2x - 15 \geq 1$, $x^2 + 2x - 16 \geq 0$. The roots of $x^2 + 2x - 16$ are $-1 \pm \sqrt{17}$, so the expression is defined for

$$x \leq -1 - \sqrt{17} \approx -5.12 \quad \text{or} \quad x \geq -1 + \sqrt{17} \approx 3.12$$

4. A sequence of points is given in the coordinate system: $A_1 (1, 1)$, $A_2 (0, 2)$, $A_3 (2, 2)$, $A_4 (4, 0)$, and we also know that if $A_n (a, b)$, then $A_{n+4} (4a, 4b)$. How long is the broken line $A_1 A_2 \dots A_{101}$?

Solution: A_{n+4} is the image of A_n under the dilation with center O and ratio 4, so every segment $A_n A_{n+5}$ is 4 times as long as $A_n A_{n+1}$. The first four segments (with $A_5 = (4, 4)$) have lengths

$$A_1 A_2 = \sqrt{2}, \quad A_2 A_3 = 2, \quad A_3 A_4 = 2\sqrt{2}, \quad A_4 A_5 = 4, \quad \text{total } 3(2 + \sqrt{2})$$

The broken line has 100 segments, that is, 25 groups of four, and each group is 4 times as long as the previous one. By the formula for the sum of a geometric sequence, the length is

$$3(2 + \sqrt{2}) \cdot \frac{4^{25} - 1}{4 - 1} = (2 + \sqrt{2})(2^{50} - 1) \approx 3.84 \cdot 10^{15}$$

5. The volume and the surface area of a regular square prism are equal (as numbers), and every edge has integer length. Find all such prisms.

Solution: Let the base edge be a and the height b . Then $a^2 b = 2a^2 + 4ab$, and dividing by a : $ab = 2a + 4b$, so

$$a = \frac{4b}{b-2} = 4 + \frac{8}{b-2}$$

So $b - 2$ is a positive divisor of 8: 1, 2, 4, or 8. The prisms are

$$a = 12, b = 3; \quad a = 8, b = 4; \quad a = 6, b = 6; \quad a = 5, b = 10$$

(the third one is a cube).

6. We add two angles of a quadrilateral (measured in radians), add the other two angles, and multiply the two sums. Show that the product is at most π^2 .

Solution: Let the two sums be u and v ; they are positive, and $u + v = 2\pi$ (the sum of the angles of a quadrilateral). By the inequality between the arithmetic and geometric means,

$$uv \leq \left(\frac{u+v}{2} \right)^2 = \pi^2$$

7. In triangle ABC , T is the foot of the altitude from C . CAQ and CBP are right triangles with the right angles at A and B , such that $AQ = BT$ and $BP = AT$. Prove that $CP = CQ$.

Solution: Let $TA = c_1$, $TB = c_2$, and $TC = m$. Since $CT \perp AB$, by the Pythagorean theorem $CA^2 = m^2 + c_1^2$ and $CB^2 = m^2 + c_2^2$. Then, again by the Pythagorean theorem,

$$CQ^2 = CA^2 + AQ^2 = m^2 + c_1^2 + c_2^2, \quad CP^2 = CB^2 + BP^2 = m^2 + c_2^2 + c_1^2$$

so $CP = CQ$.

8. Find the values of the parameter p for which the following equation has no real root.

$$\sin^4 x + \cos^4 x = p$$

Solution:

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 1 - \frac{\sin^2 2x}{2}$$

Since $0 \leq \sin^2 2x \leq 1$, the left-hand side takes exactly the values from $\frac{1}{2}$ to 1. So the equation has no real root for

$$p < \frac{1}{2} \quad \text{or} \quad p > 1$$