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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the exact value of the following expression.

$$\frac{(\log_9 27 - \log_4 2) \left(\sqrt{27} - 3\sqrt{8} - \frac{1}{3}\sqrt{243} \right)}{\sin \frac{7\pi}{4} - \cos \frac{7\pi}{4}}$$

Solution: $\log_9 27 = \frac{3}{2}$ and $\log_4 2 = \frac{1}{2}$, so the first factor is 1. $\sqrt{27} = 3\sqrt{3}$, $3\sqrt{8} = 6\sqrt{2}$, and $\frac{1}{3}\sqrt{243} = \frac{1}{3} \cdot 9\sqrt{3} = 3\sqrt{3}$, so the second factor is $-6\sqrt{2}$. The denominator is $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$. So the value is $\frac{-6\sqrt{2}}{-\sqrt{2}} = 6$.

2. The area of a right trapezoid circumscribed around a circle of radius $r = 2$ units is 18 square units. Find the lengths of the sides of the trapezoid.

Solution: (A right trapezoid has a leg perpendicular to the bases.) The perpendicular leg is the height, which is the diameter of the circle: 4. The area is $\frac{a+c}{2} \cdot 4 = 18$, so the sum of the bases is $a + c = 9$. The trapezoid has an inscribed circle, so the sums of the opposite sides are equal: the legs also add up to 9, and the slanted leg is $9 - 4 = 5$. The difference of the bases is the projection of the slanted leg: $\sqrt{5^2 - 4^2} = 3$. So the bases are 6 and 3, and the sides of the trapezoid are

6, 4, 3, 5

3. Solve the equation

$$10 \cos^2 x - 16 \sin x = \cos 2x + 15$$

over the real numbers.

Solution: Let $u = \sin x$; then $\cos^2 x = 1 - u^2$ and $\cos 2x = 1 - 2u^2$, so

$$10 - 10u^2 - 16u = 16 - 2u^2, \quad 4u^2 + 8u + 3 = 0, \quad u = -\frac{1}{2} \text{ or } u = -\frac{3}{2}$$

$\sin x = -\frac{3}{2}$ is impossible, and $\sin x = -\frac{1}{2}$ gives

$$x = -\frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = -\frac{5\pi}{6} + 2k\pi, \quad k \text{ is an integer}$$

4. Find the equation of the circle that passes through the points $A(1, 2)$ and $B(4, 5)$, and whose center is on the line $2x - y = 6$.

Solution: The center is also on the perpendicular bisector of AB : $(x - 1)^2 + (y - 2)^2 = (x - 4)^2 + (y - 5)^2$, that is, $x + y = 6$. With $2x - y = 6$ this gives the center $K(4, 2)$, and the radius is $KA = 3$. The circle is

$$(x - 4)^2 + (y - 2)^2 = 9$$

5. Solve the equation

$$\sqrt{2 + \log_x 4} = -\frac{\sqrt{12}}{\log_2 x}$$

over the real numbers.

Solution: We need $x > 0$ and $x \neq 1$. Let $y = \log_2 x \neq 0$. The right-hand side must be nonnegative, so $y < 0$, that is, $0 < x < 1$. Since $\log_x 4 = \frac{2}{\log_2 x} = \frac{2}{y}$, squaring gives

$$2 + \frac{2}{y} = \frac{12}{y^2}, \quad y^2 + y - 6 = 0, \quad y = 2 \text{ or } y = -3$$

Only $y = -3$ is negative, so $x = 2^{-3} = \frac{1}{8}$. (Check: $\sqrt{2 - \frac{2}{3}} = \frac{2}{\sqrt{3}}$ and $-\frac{\sqrt{12}}{-3} = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$.)

6. The tangents of the angles of a triangle are in the ratio $2 : 3 : 4$. What is the ratio of the sides of the triangle?

Solution: Let $\tan \alpha = 2t$, $\tan \beta = 3t$, $\tan \gamma = 4t$. In a triangle, $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$ (this follows from $\tan \gamma = -\tan(\alpha + \beta)$). So $9t = 24t^3$, and $t^2 = \frac{3}{8}$. Here $t > 0$, since a triangle cannot have three obtuse angles. Then

$$\tan^2 \alpha = \frac{3}{2}, \quad \tan^2 \beta = \frac{27}{8}, \quad \tan^2 \gamma = 6$$

and using $\sin^2 \varphi = \frac{\tan^2 \varphi}{1 + \tan^2 \varphi}$:

$$\sin^2 \alpha = \frac{3}{5}, \quad \sin^2 \beta = \frac{27}{35}, \quad \sin^2 \gamma = \frac{6}{7}$$

By the Law of Sines, $a^2 : b^2 : c^2 = \frac{3}{5} : \frac{27}{35} : \frac{6}{7} = 21 : 27 : 30 = 7 : 9 : 10$, so

$$a : b : c = \sqrt{7} : 3 : \sqrt{10}$$

7. On the side AB of the square $ABCD$ as hypotenuse, we draw right triangles outward. Prove that the bisectors of their right angles all go through one point and bisect the area of the square.

Solution: Let ABP be such a right triangle, with the right angle at P . By Thales' theorem, P is on the circle with diameter AB , on the side of the line AB opposite to the square. The bisector of the angle APB meets this circle again at a point K ; since $\angle APK = \angle BPK$, the inscribed angle theorem gives $KA = KB$, so K is the midpoint of the arc AB on the side of the square. Then ABK is an isosceles right triangle with hypotenuse AB on the side of the square, so K is the center O of the square (the diagonals of a square meet at a right angle and are equal). So every such bisector goes through O . A line through the center of a square divides it into two congruent parts (the square is symmetric about its center), so these bisectors bisect the area of the square.

8. Find all values of the real parameter p for which the inequality

$$4x^2 + 4y^2 + pxy - x - y + \frac{1}{16} \geq 0$$

holds for every pair of real numbers (x, y) with $x^2 = y^2$.

Solution: $x^2 = y^2$ means $y = x$ or $y = -x$.

If $y = -x$: the left-hand side is $(8 - p)x^2 + \frac{1}{16}$, which is nonnegative for every x if and only if $p \leq 8$.

If $y = x$: the left-hand side is $(8 + p)x^2 - 2x + \frac{1}{16}$. This is nonnegative for every x if and only if the leading coefficient is positive and the discriminant is not positive:

$$8 + p > 0, \quad 4 - 4 \cdot \frac{8 + p}{16} \leq 0, \quad \text{that is, } p \geq 8$$

So the only value is $p = 8$. (Then for $y = x$ the left-hand side is $16x^2 - 2x + \frac{1}{16} = \left(4x - \frac{1}{4}\right)^2 \geq 0$.)