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translated from: KoMaL, January 1992, pages 6-7

Pdf file: KoMaL, January 1992, pages 6-7

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The medians to the legs of a right triangle are 3 and 4 units long. Find the length of the median to the hypotenuse.

Solution: Let the legs be $2a$ and $2b$. The median to the leg $2a$ goes from the opposite acute vertex to the midpoint of that leg, so by the Pythagorean theorem its square is $a^2 + (2b)^2$; similarly for the other one:

$$a^2 + 4b^2 = 9, \quad 4a^2 + b^2 = 16$$

Adding: $5(a^2 + b^2) = 25$, so $a^2 + b^2 = 5$. The hypotenuse is $2\sqrt{a^2 + b^2}$, and the median to the hypotenuse is half of it (by Thales' theorem), so it is $\sqrt{a^2 + b^2} = \sqrt{5}$ units long. (We did not even need the legs.)

2. A circle through the vertices A and C of triangle ABC intersects the side AB at D and the side BC at E , so that

$$CE = 1, \quad CD = BE = 4, \quad \text{and} \quad AD = 4BD$$

Find the area of the triangle.

Solution: By the power of the point B with respect to the circle,

$$BD \cdot BA = BE \cdot BC, \quad BD \cdot 5BD = 4 \cdot 5, \quad BD = 2$$

so $BA = 10$ and $BC = 5$. In triangle DBC , the sides are $BD = 2$, $BC = 5$, $CD = 4$, so by the Law of Cosines

$$\cos \beta = \frac{4 + 25 - 16}{2 \cdot 2 \cdot 5} = \frac{13}{20}, \quad \sin \beta = \frac{\sqrt{231}}{20}$$

The area of triangle ABC is

$$\frac{1}{2} \cdot BA \cdot BC \cdot \sin \beta = \frac{1}{2} \cdot 10 \cdot 5 \cdot \frac{\sqrt{231}}{20} = \frac{5\sqrt{231}}{4} \approx 19.0$$

3. Solve the following inequality over the real numbers.

$$\log_{x+1} (2x^2 - 3x + 1) \leq 2$$

Solution: We need $x + 1 > 0$, $x + 1 \neq 1$, and $2x^2 - 3x + 1 = (2x - 1)(x - 1) > 0$: that is, $x > -1$, $x \neq 0$, and $x < \frac{1}{2}$ or $x > 1$. Since $2 = \log_{x+1} (x + 1)^2$:

If $x > 0$ (base greater than 1): $2x^2 - 3x + 1 \leq (x + 1)^2$, that is, $x^2 - 5x \leq 0$, $0 \leq x \leq 5$. With the conditions: $0 < x < \frac{1}{2}$ or $1 < x \leq 5$.

If $-1 < x < 0$ (base between 0 and 1): $2x^2 - 3x + 1 \geq (x + 1)^2$, that is, $x \leq 0$ or $x \geq 5$, so every $-1 < x < 0$ works.

The solution is

$$-1 < x < 0, \quad \text{or} \quad 0 < x < \frac{1}{2}, \quad \text{or} \quad 1 < x \leq 5$$

4. Solve the following equation over the real numbers.

$$\sqrt{16 \cos^4 \frac{x}{2} - 7 \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = 2 \cos x + 1$$

Solution: With $c = \cos^2 \frac{x}{2}$ and $\sin^2 \frac{x}{2} = 1 - c$, the expression under the square root is

$$16c^2 - 8c + 1 = (4c - 1)^2 = (2(1 + \cos x) - 1)^2 = (2 \cos x + 1)^2$$

(using $2 \cos^2 \frac{x}{2} = 1 + \cos x$). So the equation is $|2 \cos x + 1| = 2 \cos x + 1$, which holds exactly when $2 \cos x + 1 \geq 0$, that is, $\cos x \geq -\frac{1}{2}$:

$$-\frac{2\pi}{3} + 2n\pi \leq x \leq \frac{2\pi}{3} + 2n\pi, \quad n \text{ is an integer}$$

5. The base of the pyramid $ABCDE$ is the rectangle $ABCD$, with $AB = 5$, $BC = 2$, $EA = 3$, $EB = 4$ units. Find the lengths of the edges EC and ED of the pyramid if the volume of the pyramid is as large as possible. What is the volume of the pyramid?

Solution: The triangle EAB has sides 3, 4, 5, so it has a right angle at E , and the distance of E from AB is $\frac{3 \cdot 4}{5} = \frac{12}{5}$. The triangle EAB is fixed, and it can turn around the line AB ; the height of the pyramid is the distance of E from the base, which is largest, $\frac{12}{5}$, when the face EAB is perpendicular to the base. Then the volume is

$$V = \frac{1}{3} \cdot 5 \cdot 2 \cdot \frac{12}{5} = 8$$

In this case AD and BC are perpendicular to the plane EAB , so EAD and EBC are right triangles:

$$ED = \sqrt{9 + 4} = \sqrt{13}, \quad EC = \sqrt{16 + 4} = 2\sqrt{5}$$

6. The line $x + y = 20$ intersects the x -axis at A and the y -axis at B , and let O be the origin. Find the equation of the line e that is parallel to the line $x = 3y$, intersects the segment OB at C and the segment AB at D , and for which the area of the quadrilateral $OADC$ is 104 square units.

Solution: The triangle OAB has area 200, so the triangle BDC cut off by e has area 96. Let $C = (0, k)$; the line e has slope $\frac{1}{3}$, so it is $y = \frac{x}{3} + k$. It meets $x + y = 20$ at D with first coordinate $d = \frac{3(20-k)}{4}$. The triangle BDC has base $BC = 20 - k$ on the y -axis and height d :

$$\frac{1}{2} (20 - k) \cdot \frac{3(20 - k)}{4} = 96, \quad (20 - k)^2 = 256, \quad k = 4$$

(the other root, $k = 36$, is not on the segment OB). So $C = (0, 4)$, $D = (12, 8)$, and the line is

$$y = \frac{x}{3} + 4, \quad \text{that is,} \quad x - 3y + 12 = 0$$

7. Solve the equation

$$(2x)^2 - 4x\sqrt{m \cdot 2^m} + 2^{m+1} + m - 2 = 0$$

over the real numbers, where m is a real parameter.

Solution: The square root requires $m \geq 0$. The equation is $4x^2 - 4\sqrt{m2^m}x + (2^{m+1} + m - 2) = 0$, and

$$\frac{D}{16} = m2^m - 2^{m+1} - m + 2 = (m - 2)2^m - (m - 2) = (m - 2)(2^m - 1)$$

For $m \geq 0$, $2^m - 1 \geq 0$, with equality only at $m = 0$.

- $m = 0$: the equation is $4x^2 = 0$, so $x = 0$.
- $0 < m < 2$: $D < 0$, no real solution.
- $m = 2$: $D = 0$, $x = \frac{\sqrt{8}}{2} = \sqrt{2}$.
- $m > 2$: two different real solutions,

$$x_{1,2} = \frac{1}{2} \left(\sqrt{m \cdot 2^m} \pm \sqrt{(m - 2)(2^m - 1)} \right)$$

8. Solve the following system of equations over the triples of real numbers.

$$\begin{aligned} x^2 &= y + z + 2 \\ y^2 &= z + x + 2 \\ z^2 &= x + y + 2 \end{aligned}$$

Solution: Subtracting the second equation from the first one: $x^2 - y^2 = y - x$, that is, $(x - y)(x + y + 1) = 0$. Similarly $(y - z)(y + z + 1) = 0$.

If $x = y = z$: $x^2 = 2x + 2$, so $x = 1 \pm \sqrt{3}$.

If $x = y$ and $y + z + 1 = 0$: $z = -1 - x$, and the first equation gives $x^2 = x + (-1 - x) + 2 = 1$, so $(x, y, z) = (1, 1, -2)$ or $(-1, -1, 0)$.

If $x + y + 1 = 0$ and $y = z$: in the same way, $(-2, 1, 1)$ or $(0, -1, -1)$.

If $x + y + 1 = 0$ and $y + z + 1 = 0$: $x = z = -1 - y$, and in the same way $(1, -2, 1)$ or $(-1, 0, -1)$.

All of these satisfy all three equations, so there are 8 solutions:

$$(1 + \sqrt{3}, 1 + \sqrt{3}, 1 + \sqrt{3}), \quad (1 - \sqrt{3}, 1 - \sqrt{3}, 1 - \sqrt{3}),$$

and the permutations of $(1, 1, -2)$ and of $(-1, -1, 0)$.