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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

There are 180 minutes to solve and present all problems.

1. Solve the following equations.

a) $\cos |x| = |\cos x|$ b) $\log |x| = |\log x|$ c) $\frac{\sin(-x)}{\sin x} = -1$

Solution: a) $\cos$ is an even function, so $\cos |x| = \cos x$, and the equation says $\cos x = |\cos x|$, that is, $\cos x \geq 0$:

$$-\frac{\pi}{2} + 2k\pi \leq x \leq \frac{\pi}{2} + 2k\pi, \quad k \text{ is an integer}$$

b) The right-hand side requires $x > 0$, and then $\log |x| = \log x$, so the equation says $\log x = |\log x|$, that is, $\log x \geq 0$: the solution is $x \geq 1$.

c) We need $\sin x \neq 0$. Since $\sin$ is an odd function, $\sin(-x) = -\sin x$, so the equation holds for every $x \neq k\pi$ (k is an integer).

2. Explain why the following is true.

$$\sin 16^\circ + \sin 28^\circ + \sin 62^\circ + \sin 74^\circ = \sqrt{2} \sin 61^\circ + \sqrt{2} \sin 73^\circ$$

Solution: By the sum-to-product formula $\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$, and $2 \sin 45^\circ = \sqrt{2}$:

$$\begin{aligned} \sin 74^\circ + \sin 16^\circ &= 2 \sin 45^\circ \cos 29^\circ = \sqrt{2} \sin 61^\circ \\ \sin 62^\circ + \sin 28^\circ &= 2 \sin 45^\circ \cos 17^\circ = \sqrt{2} \sin 73^\circ \end{aligned}$$

Adding these gives the statement.

3. The equation of the midline of triangle ABC parallel to the side AB is k : $x - 2y + 6 = 0$, its centroid is $S\left(\frac{5}{3}, \frac{4}{3}\right)$, the vertex C is $(-1, 10)$, and another vertex is on the x -axis. Find the coordinates of the missing vertices.

Solution: The midline is halfway between C and the line AB , and parallel to AB . The expression $x - 2y + 6$ is -15 at C , so it is $+15$ on the line AB : the line AB is $x - 2y - 9 = 0$. Its point on the x -axis is $(9, 0)$. The centroid is the average of the vertices, so the third vertex is

$$3S - (9, 0) - (-1, 10) = (5 - 9 + 1, 4 - 0 - 10) = (-3, -6)$$

(which is indeed on the line $x - 2y - 9 = 0$). So the missing vertices are $(9, 0)$ and $(-3, -6)$.

4. Prove that

$$\frac{1}{p \cdot q} = \frac{1}{a^2} + \frac{1}{b^2}$$

where a and b are the legs of a right triangle, and p and q are the projections of the two legs onto the hypotenuse.

Solution: Let $c = p + q$ be the hypotenuse. By the leg theorem (from similar triangles), $a^2 = pc$ and $b^2 = qc$, so $a^2b^2 = pqc^2$. Then, using $a^2 + b^2 = c^2$,

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{a^2b^2} = \frac{c^2}{pqc^2} = \frac{1}{pq}$$

5. On the sides of triangle ABC , we draw similar plane figures, whose areas are t_a , t_b , and t_c . Prove that $t_b + t_c - t_a = 2 \cdot \sqrt{t_b \cdot t_c} \cdot \cos \alpha$, where α is the interior angle of the triangle opposite the side of t_a .

Solution: The areas of similar figures are proportional to the squares of corresponding sides, so there is a positive number k with $t_a = ka^2$, $t_b = kb^2$, $t_c = kc^2$. By the Law of Cosines, $b^2 + c^2 - a^2 = 2bc \cos \alpha$. Multiplying by k :

$$t_b + t_c - t_a = 2kbc \cos \alpha = 2\sqrt{kb^2 \cdot kc^2} \cos \alpha = 2\sqrt{t_b t_c} \cos \alpha$$

6. Solve the following system of equations over the real numbers.

$$\begin{aligned} x + y &= 1 \\ x^5 + y^5 &= 2 \end{aligned}$$

Solution: Let $s = x + y = 1$ and $p = xy$. Then

$$x^5 + y^5 = s^5 - 5s^3p + 5sp^2 = 1 - 5p + 5p^2$$

(this can be checked by expanding $(x + y)^5$). So $5p^2 - 5p - 1 = 0$, and $p = \frac{5 \pm 3\sqrt{5}}{10}$. The numbers x and y are the roots of $t^2 - t + p = 0$, which are real only if $1 - 4p \geq 0$. For $p = \frac{5 + 3\sqrt{5}}{10} \approx 1.17$ this fails. For $p = \frac{5 - 3\sqrt{5}}{10} \approx -0.17$:

$$1 - 4p = \frac{6\sqrt{5} - 5}{5}, \quad x, y = \frac{1}{2} \pm \sqrt{\frac{6\sqrt{5} - 5}{20}}$$

So the solutions are $(x, y) \approx (1.149, -0.149)$ and $(-0.149, 1.149)$, exactly

$$x = \frac{1}{2} + \sqrt{\frac{6\sqrt{5} - 5}{20}}, y = \frac{1}{2} - \sqrt{\frac{6\sqrt{5} - 5}{20}} \quad \text{and} \quad x = \frac{1}{2} - \sqrt{\frac{6\sqrt{5} - 5}{20}}, y = \frac{1}{2} + \sqrt{\frac{6\sqrt{5} - 5}{20}}$$

7. A cone standing on its apex (with base radius r and height m) is filled with water up to a certain height, so that when the cone is turned onto its base, the water is half as high as the cone. How high was the water?

Solution: When the cone stands on its base, the water fills it up to height $\frac{m}{2}$, and the empty part at the top is a cone similar to the whole one with ratio $\frac{1}{2}$, whose volume is $\frac{1}{8}$ of the volume V of the cone. So the volume of the water is $\frac{7}{8}V$. When the cone stands on its apex, the water fills a cone of height x similar to the whole one, whose volume is $\left(\frac{x}{m}\right)^3 V$. So

$$\left(\frac{x}{m}\right)^3 = \frac{7}{8}, \quad x = \frac{\sqrt[3]{7}}{2}m \approx 0.956m$$

(The answer does not depend on r .)

Note: The solution sketch published in KoMaL uses $r_1^2 \pi x$ for the volume of the water cone, leaving out the factor $\frac{1}{3}$, and gets the incorrect answer $m \sqrt[3]{7/24} \approx 0.663m$.

8. For a non-constant arithmetic sequence, dividing the sum of the first n terms by the sum of the next n terms gives the same quotient for every n . What is this quotient?

Solution: Let the first term be a and the difference $d \neq 0$. The sum of the first n terms is $\frac{n}{2}(2a + (n-1)d)$, and the sum of the next n terms is $\frac{n}{2}(2a + (3n-1)d)$ (starting from $a + nd$). So for every n

$$\frac{2a - d + nd}{2a - d + 3nd} = K, \quad (2a - d) + nd = K(2a - d) + 3Knd$$

Since this holds for every n , the coefficients of n are equal: $d = 3Kd$, so $K = \frac{1}{3}$; and then the constant terms give $2a - d = \frac{1}{3}(2a - d)$, so $d = 2a$. The quotient is $\frac{1}{3}$. (For example, for 1, 3, 5, ..., the sums are n^2 and $3n^2$.)