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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. How many sides does the regular polygon have in which the measure of an exterior angle in degrees is 18 less than the number of sides?

Solution: The exterior angles of a regular n -gon are $\frac{360^\circ}{n}$. So

$$\frac{360}{n} = n - 18, \quad n^2 - 18n - 360 = 0, \quad (n - 30)(n + 12) = 0$$

Since n is a positive integer, $n = 30$: the polygon has 30 sides.

2. For which real numbers x are the following expressions negative?

$$\text{a) } \frac{x^4 - 6x^2 + 8}{4 - x^2} \quad \text{b) } \log_{\frac{1}{2}} \cos x - 1 \quad \text{c) } |x - 2| - |x + 2| + x$$

Solution: a) $x^4 - 6x^2 + 8 = (x^2 - 2)(x^2 - 4)$, so for $x \neq \pm 2$ the expression is $2 - x^2$. It is negative if $x^2 > 2$: for $x < -2$, $-2 < x < -\sqrt{2}$, $\sqrt{2} < x < 2$, and $x > 2$.

b) We need $\log_{\frac{1}{2}} \cos x < 1 = \log_{\frac{1}{2}} \frac{1}{2}$. Since $\log_{\frac{1}{2}}$ is decreasing, this means $\cos x > \frac{1}{2}$:

$$-\frac{\pi}{3} + 2k\pi < x < \frac{\pi}{3} + 2k\pi, \quad k \text{ is an integer}$$

c) Without the absolute values:

$$|x - 2| - |x + 2| + x = \begin{cases} x - 4 & \text{if } x \geq 2 \\ -x & \text{if } -2 < x < 2 \\ x + 4 & \text{if } x \leq -2 \end{cases}$$

So the expression is negative for $x < -4$ and for $0 < x < 4$.

3. The fifth term of an arithmetic sequence is 35, and the fiftieth term is 260. Leave out every term of the sequence whose last digit is 5. Express, in terms of n , the quotient of the sums of the first n terms of the left out sequence and of the remaining sequence (n is a positive integer). For which values of n is this quotient greater than $1 - \frac{1}{1003}$?

Solution: $a_1 + 4d = 35$ and $a_1 + 49d = 260$, so $d = 5$ and $a_1 = 15$: the sequence is 15, 20, 25, 30, ... The terms ending in 5 are 15, 25, 35, ... (first term 15, difference 10), and the remaining terms are 20, 30, 40, ... (first term 20, difference 10). The sums of the first n terms are

$$S_1 = \frac{n}{2} (30 + 10(n-1)) = 5n(n+2), \quad S_2 = \frac{n}{2} (40 + 10(n-1)) = 5n(n+3)$$

so the quotient is $\frac{S_1}{S_2} = \frac{n+2}{n+3} = 1 - \frac{1}{n+3}$. It is greater than $1 - \frac{1}{1003}$ if $\frac{1}{n+3} < \frac{1}{1003}$, that is, for $n > 1000$.

4. Three vertices of a trapezoid $ABCD$ are $A(2, -4)$, $B(-4, 0)$, $C(-1, 1)$. AB is one of the parallel sides of the trapezoid, and $BC = AD$. Find the coordinates of the fourth vertex of the trapezoid.

Solution: $\overrightarrow{AB} = (-6, 4)$, so the line through C parallel to AB is $2x + 3y = 1$, and D is on it. Also $AD = BC = \sqrt{9+1} = \sqrt{10}$, so D is on the circle $(x-2)^2 + (y+4)^2 = 10$. Substituting $x = \frac{1-3y}{2}$:

$$\frac{9(1+y)^2}{4} + (y+4)^2 = 10, \quad 13y^2 + 50y + 33 = 0, \quad y = -3 \text{ or } y = -\frac{11}{13}$$

So $D_1(5, -3)$ or $D_2\left(\frac{23}{13}, -\frac{11}{13}\right)$. With D_1 , $ABCD_1$ is a parallelogram (which counts as a trapezoid if a trapezoid is defined as a quadrilateral with at least one pair of parallel sides); with D_2 , $ABCD_2$ is an isosceles trapezoid.

5. Solve the following system of equations over the real numbers.

$$\begin{aligned} x + 2y &= 1 \\ x^3y + 4xy^3 &= -410 \end{aligned}$$

Solution: The second equation is $xy(x^2 + 4y^2) = -410$. Squaring the first one: $x^2 + 4y^2 + 4xy = 1$. Let $z = xy$ and $u = x^2 + 4y^2 \geq 0$. Then $u = 1 - 4z$ and $zu = -410$:

$$z(1-4z) = -410, \quad 4z^2 - z - 410 = 0, \quad z = -10 \text{ or } z = \frac{41}{4}$$

$z = \frac{41}{4}$ would give $u = -40 < 0$, which is impossible. So $xy = -10$ and $x + 2y = 1$: $(1-2y)y = -10$, $2y^2 - y - 10 = 0$, so $y = -2$ or $y = \frac{5}{2}$. The solutions are

$$(x, y) = (5, -2) \quad \text{and} \quad (x, y) = \left(-4, \frac{5}{2}\right)$$

6. Find the ratio of the surface areas of a right circular cone and the sphere inscribed in it, if the ratio of their volumes is 3 : 1.

Solution: The ratio of the surface areas is also 3 : 1.

Let r be the radius of the sphere and O its center. Cut the cone into thin pieces from O : each piece is like a pyramid (or cone) with apex O , whose base is a small part of the surface of the cone, and whose height is r (the surface of the cone is tangent to the sphere everywhere). So the volume of the cone is $V = \frac{1}{3}rA$, where A is its surface area. For the sphere, similarly, $V_{\text{sphere}} = \frac{4}{3}r^3\pi = \frac{1}{3}r \cdot 4r^2\pi$. So

$$\frac{A}{A_{\text{sphere}}} = \frac{V}{V_{\text{sphere}}} = 3$$

(With a computation: let R be the radius of the base, m the height, and a the slant height. The volume condition is $R^2m = 12r^3$, and in the cross-section through the axis, the area of the triangle is $Rm = r(a + R)$. So $\frac{R\pi(R + a)}{4r^2\pi} = \frac{R^2m}{4r^3} = \frac{12r^3}{4r^3} = 3$.)

7. Find the sides of a right triangle if they are natural numbers, and the perimeter and the area of the triangle are equal (as numbers).

Solution: Let the legs be a, b and the hypotenuse c . Then $c = \frac{ab}{2} - a - b$, and $c^2 = a^2 + b^2$:

$$\left(\frac{ab}{2} - a - b\right)^2 = a^2 + b^2, \quad \frac{a^2b^2}{4} - ab(a + b) + 2ab = 0$$

Dividing by ab and multiplying by 4: $ab - 4a - 4b + 8 = 0$, that is, $(a - 4)(b - 4) = 8$. If both factors were negative, then $4 - a$ and $4 - b$ would be positive integers at most 3, and 8 is not such a product. So we may assume $a - 4 = 8$, $b - 4 = 1$ or $a - 4 = 4$, $b - 4 = 2$: $(a, b) = (12, 5)$ or $(8, 6)$. There are two such triangles:

$$5, 12, 13 \quad \text{and} \quad 6, 8, 10$$

(Check: $5 + 12 + 13 = 30 = \frac{5 \cdot 12}{2}$ and $6 + 8 + 10 = 24 = \frac{6 \cdot 8}{2}$.)

8. Consider the function $x \mapsto (1 - p)x^2 - 2(2p + 1)x - 4p - 1$, where x is any real number and p is a real parameter. Find the value of p for which the largest value of the function is 8.

Solution: If $p = 1$, the function is linear and has no largest value. Otherwise it is quadratic, and it has a largest value only if the leading coefficient is negative: $p > 1$. Then the largest value is at $x_0 = \frac{2p + 1}{1 - p}$, and it is

$$-\frac{(2p + 1)^2}{1 - p} - 4p - 1 = 8$$

Multiplying by $1 - p$: $-(2p + 1)^2 - (4p + 1)(1 - p) = 8(1 - p)$, that is, $-7p - 2 = 8 - 8p$, so $p = 10$. Then the function is $-9x^2 - 42x - 41$, and its largest value is 8, taken at $x_0 = -\frac{7}{3}$.