

author: Imre Rábai

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned} x^2y + xy^2 &= -12 \\ \frac{1}{x} + \frac{1}{y} &= -\frac{4}{3} \end{aligned}$$

Solution: We need $x \neq 0, y \neq 0$. The equations say $xy(x+y) = -12$ and $\frac{x+y}{xy} = -\frac{4}{3}$, that is, $x+y = -\frac{4}{3}xy$.

Substituting: $-\frac{4}{3}(xy)^2 = -12$, so $(xy)^2 = 9$.

If $xy = 3$: $x+y = -4$, so x and y are the roots of $t^2 + 4t + 3 = 0$: -1 and -3 .

If $xy = -3$: $x+y = 4$, so x and y are the roots of $t^2 - 4t - 3 = 0$: $2 \pm \sqrt{7}$.

The solutions are $(-1, -3), (-3, -1), (2 + \sqrt{7}, 2 - \sqrt{7}), (2 - \sqrt{7}, 2 + \sqrt{7})$.

2. The parallel sides of an isosceles trapezoid are 8 and 48 units long. Its leg is equal to the sum of the shorter parallel side and the height. Find the area of the trapezoid and the radius of its circumscribed circle.

Solution: Let m be the height; the leg is $m + 8$, and its projection onto the longer base is $\frac{48-8}{2} = 20$. So

$$(m+8)^2 = m^2 + 400, \quad 16m = 336, \quad m = 21$$

and the area is $\frac{8+48}{2} \cdot 21 = 588$ square units.

Put the longer base on the x -axis from $(-24, 0)$ to $(24, 0)$; the shorter base is from $(-4, 21)$ to $(4, 21)$. The center of the circumscribed circle is on the axis of symmetry, $(0, k)$, equally far from $(24, 0)$ and $(4, 21)$:

$$576 + k^2 = 16 + (21 - k)^2, \quad 42k = -119, \quad k = -\frac{17}{6}$$

(so the center is outside the trapezoid, below the longer base), and the radius is $r = \sqrt{576 + \frac{289}{36}} = \sqrt{\frac{21025}{36}} = \frac{145}{6}$.

3. Solve the following inequality over the real numbers.

$$3 \cdot 4^{x+1} - 35 \cdot 6^x + 2 \cdot 9^{x+1} \geq 0$$

Solution: The inequality is $12 \cdot 4^x - 35 \cdot 6^x + 18 \cdot 9^x \geq 0$. Dividing by $9^x > 0$ and writing $t = \left(\frac{2}{3}\right)^x > 0$:

$$12t^2 - 35t + 18 \geq 0, \quad (3t - 2)(4t - 9) \geq 0, \quad t \leq \frac{2}{3} \text{ or } t \geq \frac{9}{4}$$

Since $\left(\frac{2}{3}\right)^x$ is decreasing, $\left(\frac{2}{3}\right)^x \leq \frac{2}{3}$ means $x \geq 1$, and $\left(\frac{2}{3}\right)^x \geq \frac{9}{4} = \left(\frac{2}{3}\right)^{-2}$ means $x \leq -2$. The solution is $x \leq -2$ or $x \geq 1$.

4. Find the equation of the circle through the point $P(9, 5)$, if it is tangent to the y -axis at a point Q for which $PQ = 3\sqrt{10}$.

Solution: Let $Q = (0, v)$. Then $81 + (5 - v)^2 = 90$, so $v = 2$ or $v = 8$. A circle tangent to the y -axis at $(0, v)$ has center (u, v) and radius $|u|$, so P on the circle means

$$(9 - u)^2 + (5 - v)^2 = u^2, \quad 81 - 18u + 9 = 0, \quad u = 5$$

There are two such circles:

$$(x - 5)^2 + (y - 2)^2 = 25 \quad \text{and} \quad (x - 5)^2 + (y - 8)^2 = 25$$

5. The sides of a triangle are 4, 6, and 8 units long. Find the radius of the circle whose center is on the longest side and which is tangent to the other two sides. Find the length of the angle bisector to the longest side.

Solution: Let $AC = 4$, $BC = 6$, $AB = 8$. The center of the circle is equally far from CA and CB , so it is the point F where the bisector of the angle at C meets AB . By the angle bisector theorem $AF : FB = 4 : 6$, so $AF = 3.2$ and $FB = 4.8$. By the Law of Cosines,

$$\cos \alpha = \frac{16 + 64 - 36}{2 \cdot 4 \cdot 8} = \frac{11}{16}, \quad \sin \alpha = \frac{3\sqrt{15}}{16}$$

and the radius is the distance of F from AC : $r = AF \sin \alpha = \frac{3\sqrt{15}}{5} \approx 2.32$ units.

The bisector CF (from the Law of Cosines in triangle ACF , or from the formula $CF^2 = AC \cdot BC - AF \cdot FB$):

$$CF^2 = 16 + 10.24 - 2 \cdot 4 \cdot 3.2 \cdot \frac{11}{16} = 8.64, \quad CF = \sqrt{8.64} = \frac{6\sqrt{6}}{5} \approx 2.94$$

6. What are the largest and smallest values of the function $x \mapsto |x^2 - x| + |x^2 + 3x + 2|$ on the closed interval $\left[-\frac{3}{2}, \frac{1}{2}\right]$?
Where does it take these values?

Solution: $x^2 - x = x(x - 1)$ and $x^2 + 3x + 2 = (x + 1)(x + 2)$. On the given interval:

$$f(x) = \begin{cases} -4x - 2 & \text{if } -\frac{3}{2} \leq x < -1 \\ 2x^2 + 2x + 2 = 2\left(x + \frac{1}{2}\right)^2 + \frac{3}{2} & \text{if } -1 \leq x \leq 0 \\ 4x + 2 & \text{if } 0 < x \leq \frac{1}{2} \end{cases}$$

On the first part $2 < f(x) \leq 4$, on the second part $\frac{3}{2} \leq f(x) \leq 2$, and on the third part $2 < f(x) \leq 4$. So the largest value is 4, at $x = -\frac{3}{2}$ and $x = \frac{1}{2}$, and the smallest value is $\frac{3}{2}$, at $x = -\frac{1}{2}$.

7. Twice one root of the equation $x^2 + px - q = 0$ is a root of the equation $x^2 - 3px + q = 0$. Find p and q if $p^2 + 4q = 36$.

Solution: Let x_1 be the root; then $x_1^2 + px_1 - q = 0$ and $4x_1^2 - 6px_1 + q = 0$. Adding: $5x_1^2 - 5px_1 = 0$, so $x_1 = 0$ or $x_1 = p$.

If $x_1 = 0$: $q = 0$, and $p^2 = 36$, so $p = \pm 6$.

If $x_1 = p$: $2p^2 - q = 0$, so $q = 2p^2$, and $9p^2 = 36$: $p = \pm 2$, $q = 8$.

The solutions are $(p, q) = (6, 0), (-6, 0), (2, 8), (-2, 8)$. (For example, for $p = 2$, $q = 8$: $x^2 + 2x - 8 = 0$ has the root 2, and 4 is a root of $x^2 - 6x + 8 = 0$.)

8. Consider the equation $p \cdot \tan x = 1 - \cos 2x$, where p is a real parameter.

a) Solve the equation if $|p| = \frac{1}{2}$.

b) Depending on the value of p , how many solutions of the equation are in the closed interval $[0, \pi]$?

Solution: We need $\cos x \neq 0$. Since $1 - \cos 2x = 2 \sin^2 x$, the equation is $\frac{p \sin x}{\cos x} = 2 \sin^2 x$, that is,

$$\sin x (p - 2 \sin x \cos x) = 0, \quad \sin x (p - \sin 2x) = 0$$

So $x = n\pi$ is always a solution, and the other solutions satisfy $\sin 2x = p$ (with $\cos x \neq 0$).

a) If $p = \frac{1}{2}$: $x = n\pi$, $x = \frac{\pi}{12} + n\pi$, $x = \frac{5\pi}{12} + n\pi$. If $p = -\frac{1}{2}$: $x = n\pi$, $x = \frac{7\pi}{12} + n\pi$, $x = \frac{11\pi}{12} + n\pi$ (n is an integer).

b) $x = 0$ and $x = \pi$ are always solutions. For $x \in [0, \pi]$, $2x$ runs over $[0, 2\pi]$.

- $|p| > 1$: $\sin 2x = p$ has no solution, so there are 2 solutions.
- $p = 0$: $\sin 2x = 0$ gives $x = 0, \frac{\pi}{2}, \pi$, but $x = \frac{\pi}{2}$ is not allowed: 2 solutions.
- $p = 1$: $x = \frac{\pi}{4}$ is added: 3 solutions. $p = -1$: $x = \frac{3\pi}{4}$ is added: 3 solutions.
- $0 < |p| < 1$: $\sin 2x = p$ has two solutions with $0 < x < \pi$, $x \neq \frac{\pi}{2}$: 4 solutions.