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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\frac{1-2x}{3x+1} - \frac{5x+1}{x-1} = \frac{24}{2x-3x^2+1}$$

Solution: $2x - 3x^2 + 1 = -(3x + 1)(x - 1)$, so we need $x \neq -\frac{1}{3}$ and $x \neq 1$. Multiplying by $(3x + 1)(x - 1)$:

$$(1 - 2x)(x - 1) - (5x + 1)(3x + 1) = -24, \quad 17x^2 + 5x - 22 = 0, \quad (x - 1)(17x + 22) = 0$$

$x = 1$ is not allowed, so the only solution is $x = -\frac{22}{17}$.

2. Prove that the value of the expression

$$\sqrt{4 \cos^4 \alpha - 6 \cos 2\alpha + 3} + \sqrt{4 \sin^4 \alpha + 6 \cos 2\alpha + 3}$$

does not depend on α .

Solution: Using $\cos 2\alpha = 2 \cos^2 \alpha - 1$ in the first term and $\cos 2\alpha = 1 - 2 \sin^2 \alpha$ in the second one:

$$4 \cos^4 \alpha - 12 \cos^2 \alpha + 9 = (2 \cos^2 \alpha - 3)^2, \quad 4 \sin^4 \alpha - 12 \sin^2 \alpha + 9 = (2 \sin^2 \alpha - 3)^2$$

Both $2 \cos^2 \alpha - 3$ and $2 \sin^2 \alpha - 3$ are negative, so the expression is

$$(3 - 2 \cos^2 \alpha) + (3 - 2 \sin^2 \alpha) = 6 - 2 = 4$$

for every α .

3. Find the values of the parameter m for which the inequality

$$3x^2 + 5x + 1 > 2x^2 + mx - 3$$

holds for every real number x .

Solution: The inequality is $x^2 + (5 - m)x + 4 > 0$. This holds for every x if and only if the discriminant is negative:

$$(5 - m)^2 - 16 < 0, \quad |5 - m| < 4, \quad 1 < m < 9$$

4. The coordinates of two opposite vertices of the rectangle $ABCD$ are $A(-6, 0)$ and $C(1, -1)$. The vertex B is on the y -axis. Find the coordinates of the vertices B and D .

Solution: Let $B = (0, h)$. The angle at B is a right angle, so $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$:

$$(-6, -h) \cdot (1, -1 - h) = -6 + h(h + 1) = 0, \quad h = 2 \text{ or } h = -3$$

Since the diagonals bisect each other, $D = A + C - B$. So $B(0, 2)$, $D(-5, -3)$, or $B(0, -3)$, $D(-5, 2)$.

5. In an arithmetic sequence $a_1 = 12$ and $d = -2$. Find the term of the sequence that is one fifth of the sum of the terms before it.

Solution: $a_n = 14 - 2n$, and the sum of the first $n - 1$ terms is $S_{n-1} = \frac{n-1}{2}(24 - 2(n-2)) = (n-1)(14-n)$.

The condition $5a_n = S_{n-1}$ gives

$$70 - 10n = -n^2 + 15n - 14, \quad n^2 - 25n + 84 = 0, \quad n = 4 \text{ or } n = 21$$

Both work: $a_4 = 6$ and $S_3 = 12 + 10 + 8 = 30 = 5 \cdot 6$; $a_{21} = -28$ and $S_{20} = -140 = 5 \cdot (-28)$.

6. For 18 months, at the beginning of every month we deposit 2000 Ft in a bank, at 30% annual compound interest.

a) What monthly compound interest rate (in percent) corresponds to 30% annual interest?

b) How much money will we have in the bank at the end of the second year?

Solution: a) If the monthly rate is $p\%$ and $q = 1 + \frac{p}{100}$, then $q^{12} = 1.3$, so $q = \sqrt[12]{1.3} \approx 1.022104$, and $p \approx 2.21$.

b) The second year ends at the end of the 24th month. The deposit made at the beginning of month k earns interest for $25 - k$ months, so the deposits of months 1, 2, ..., 18 are worth $2000q^{24}$, $2000q^{23}$, ..., $2000q^7$. In total:

$$2000q^7 + 2000q^8 + \cdots + 2000q^{24} = 2000q^7 \cdot \frac{q^{18} - 1}{q - 1} \approx 50\,848 \text{ Ft}$$

7. Solve the equation $\tan\left(x + \frac{3\pi}{4}\right) = \tan 2x + 1$ over the real numbers.

Solution: We need $x + \frac{3\pi}{4} \neq \frac{\pi}{2} + n\pi$ and $2x \neq \frac{\pi}{2} + n\pi$.

If $x = \frac{\pi}{2} + k\pi$: the left-hand side is $\tan\left(\frac{5\pi}{4} + k\pi\right) = 1$ and the right-hand side is $\tan(\pi + 2k\pi) + 1 = 1$, so these are solutions.

Otherwise $\tan x = t$ is defined, and $\tan\left(x + \frac{3\pi}{4}\right) = \frac{t-1}{1+t}$, $\tan 2x = \frac{2t}{1-t^2}$ (with $t \neq \pm 1$). Multiplying by $1 - t^2 = (1-t)(1+t)$:

$$(t-1)(1-t) = 2t + 1 - t^2, \quad -t^2 + 2t - 1 = 2t + 1 - t^2, \quad -1 = 1$$

which is impossible. So the solutions are

$$x = \frac{\pi}{2} + k\pi, \quad k \text{ is an integer}$$

8. Prove that if $x + y = 1$, $x > 0$, and $y > 0$, then

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}$$

When are the two sides equal?

Solution: By the inequality between the arithmetic and geometric means, $\sqrt{xy} \leq \frac{x+y}{2} = \frac{1}{2}$, so $xy \leq \frac{1}{4}$ and $\frac{1}{x^2y^2} \geq 16$. Also $x^2 + y^2 = (x+y)^2 - 2xy = 1 - 2xy \geq \frac{1}{2}$. So

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 = x^2 + y^2 + 4 + \frac{1}{x^2} + \frac{1}{y^2} = x^2 + y^2 + 4 + \frac{x^2 + y^2}{x^2y^2} \geq \frac{1}{2} + 4 + 16 \cdot \frac{1}{2} = \frac{25}{2}$$

Equality holds if and only if $x = y = \frac{1}{2}$.