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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

a) $(2x^2 - x - 1)(x^2 - 1) = 0$ b) $(2x^2 - x - 1)^2 + (x^2 - 1)^2 = 0$

Solution: $2x^2 - x - 1 = (2x + 1)(x - 1)$ and $x^2 - 1 = (x + 1)(x - 1)$.

a) A product is zero if one of the factors is zero: $x = 1$, $x = -\frac{1}{2}$, or $x = -1$.

b) A sum of two squares is zero only if both are zero, so both factors must be zero: the only solution is $x = 1$.

2. Prove the identity

$$\frac{1 + \tan \alpha}{1 - \tan \alpha} = \frac{1 + \sin 2\alpha}{\cos 2\alpha}$$

For which real numbers α is the identity true?

Solution: The left-hand side is defined if $\tan \alpha$ is defined and $\tan \alpha \neq 1$, and the right-hand side if $\cos 2\alpha \neq 0$. So we need $\alpha \neq \frac{\pi}{2} + n\pi$ and $\alpha \neq \frac{\pi}{4} + n \cdot \frac{\pi}{2}$. For these α ,

$$\frac{1 + \sin 2\alpha}{\cos 2\alpha} = \frac{(\cos \alpha + \sin \alpha)^2}{(\cos \alpha + \sin \alpha)(\cos \alpha - \sin \alpha)} = \frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} = \frac{1 + \tan \alpha}{1 - \tan \alpha}$$

(dividing the numerator and the denominator by $\cos \alpha$ at the end). So the identity holds for every real α except $\frac{\pi}{2} + n\pi$ and $\frac{\pi}{4} + n \cdot \frac{\pi}{2}$ (n is an integer).

3. Solve the equation $x^{\log_3 x} = 9x$ over the real numbers.

Solution: We need $x > 0$. Taking the base 3 logarithm of both sides: $(\log_3 x)^2 = 2 + \log_3 x$, so $\log_3 x = 2$ or $\log_3 x = -1$. The solutions are $x = 9$ and $x = \frac{1}{3}$.

4. Two sides of a triangle are 3 and 4 units long, and its area is $3\sqrt{2}$ square units. Find the third side of the triangle.

Solution: Let γ be the angle between the two given sides. Then $\frac{3 \cdot 4 \sin \gamma}{2} = 3\sqrt{2}$, so $\sin \gamma = \frac{\sqrt{2}}{2}$, and $\gamma = 45^\circ$ or 135° . By the Law of Cosines, the third side is

$$\sqrt{25 - 12\sqrt{2}} \approx 2.83 \quad \text{or} \quad \sqrt{25 + 12\sqrt{2}} \approx 6.48$$

5. The orthocenter of triangle ABC is the origin, and $C(0, -7)$. Find the coordinates of the missing vertices and the area of the triangle, if the side AB goes through the point $P(0, 9)$ and its length is as short as possible.

Solution: The altitude from C goes through the orthocenter, so it is the y -axis, and AB is perpendicular to it: AB is the line $y = 9$. Let $A(a, 9)$ and $B(-b, 9)$. The altitude from B goes through the origin, so $\overrightarrow{OB} \perp \overrightarrow{CA}$:

$$(-b, 9) \cdot (a, 16) = -ab + 144 = 0, \quad ab = 144$$

So a and b have the same sign; we may assume they are positive (otherwise exchange A and B), and $AB = a + b$. By the inequality between the arithmetic and geometric means, $a + b \geq 2\sqrt{ab} = 24$, with equality for $a = b = 12$. So $A(12, 9)$, $B(-12, 9)$, and the area of the triangle is $\frac{24 \cdot 16}{2} = 192$ square units.

6. In rectangle $ABCD$ the side AB is twice the side AD . The distances of a point P of the plane from the vertices A , B , and D are $PA = 2$, $PB = 4$, $PD = 3$ units. Find the area of the rectangle. Where is the point P ?

Solution: Use coordinates: $A(0, 0)$, $B(2a, 0)$, $D(0, a)$, $P(x, y)$. Then

$$x^2 + y^2 = 4, \quad (2a - x)^2 + y^2 = 16, \quad x^2 + (a - y)^2 = 9$$

Subtracting the first equation from the other two: $x = \frac{a^2 - 3}{a}$ and $y = \frac{a^2 - 5}{2a}$. Substituting into the first one and multiplying by $4a^2$:

$$4(a^2 - 3)^2 + (a^2 - 5)^2 = 16a^2, \quad 5a^4 - 50a^2 + 61 = 0, \quad a^2 = 5 \pm \frac{8\sqrt{5}}{5}$$

The area of the rectangle is $2a^2$:

$$10 + \frac{16\sqrt{5}}{5} \approx 17.16 \quad \text{or} \quad 10 - \frac{16\sqrt{5}}{5} \approx 2.84$$

In the first case $x \approx 1.90 > 0$ and $y \approx 0.61 > 0$ (and $x < 2a$, $y < a$), so P is inside the rectangle. In the second case $x \approx -1.32 < 0$ and $y \approx -1.50 < 0$, so P is outside the rectangle.

Note: In KoMaL the last question is printed as “Where is the point D ?”, which is most likely a misprint for P .

7. Find the value of the parameter p for which the largest value of the polynomial

$$f(x) = -x^2 + 4px - 4p^2 + 2p + 2$$

on the interval $[0, 3]$ is -2 .

Solution: $f(x) = -(x - 2p)^2 + 2p + 2$, a downward parabola with vertex at $x = 2p$.

If $2p < 0$: f is decreasing on $[0, 3]$, and its largest value is $f(0) = -4p^2 + 2p + 2 = -2$, that is, $2p^2 - p - 2 = 0$.

The negative root is $p = \frac{1 - \sqrt{17}}{4} \approx -0.78$.

If $0 \leq 2p \leq 3$: the largest value is $f(2p) = 2p + 2 = -2$, so $p = -2$, which is not in this range.

If $2p > 3$: f is increasing on $[0, 3]$, and its largest value is $f(3) = -9 + 12p - 4p^2 + 2p + 2 = -2$, that is, $4p^2 - 14p + 5 = 0$. The root greater than $\frac{3}{2}$ is $p = \frac{7 + \sqrt{29}}{4} \approx 3.10$.

So $p = \frac{1 - \sqrt{17}}{4}$ or $p = \frac{7 + \sqrt{29}}{4}$.

Note: In KoMaL the polynomial is called $p(x)$, the same letter as the parameter, so we call it $f(x)$. The solution sketch in KoMaL gives $\frac{1 - \sqrt{17}}{2}$ for the first value; that is a slip, the correct value is $\frac{1 - \sqrt{17}}{4}$.

8. The difference of two positive integers is k . If we add 3 times the larger number to 4 times the smaller number, we get 84. For which values of k does the problem have a solution, and what are the solutions?

Solution: Let the numbers be $x > y > 0$, with $x = y + k$. Then $3(y + k) + 4y = 84$, that is, $7y = 84 - 3k$, $y = 12 - \frac{3k}{7}$. So k must be a multiple of 7, and $y > 0$ requires $k < 28$. The solutions:

$$k = 7 : 16, 9; \quad k = 14 : 20, 6; \quad k = 21 : 24, 3$$