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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets usually consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The roots of the equation $x^2 - 16x + 57 = 0$ are a and b . Without computing a and b , find the sum of the volumes of the cone with height a and base radius b and the cone with height b and base radius a .

Solution: The discriminant is $256 - 228 > 0$, and the roots have positive sum and product, so a and b are positive. By Viète's formulas $a + b = 16$ and $ab = 57$. The sum of the volumes is

$$\frac{\pi}{3}b^2a + \frac{\pi}{3}a^2b = \frac{\pi}{3}ab(a + b) = \frac{\pi}{3} \cdot 57 \cdot 16 = 304\pi$$

2. The radius of the inscribed circle of a triangle is 4 units, and the radii of the excircles are 14, 12, and 10.5 units. Find the sides and the area of the triangle.

Solution: Let t be the area and s the semiperimeter. The known formulas are

$$t = rs = r_a(s - a) = r_b(s - b) = r_c(s - c)$$

Multiplying these four equations and using Heron's formula $s(s - a)(s - b)(s - c) = t^2$:

$$t^4 = rr_ar_br_c \cdot t^2, \quad t^2 = 4 \cdot 14 \cdot 12 \cdot 10.5 = 7056, \quad t = 84$$

Then $s = \frac{84}{4} = 21$, and $s - a = \frac{84}{14} = 6$, $s - b = \frac{84}{12} = 7$, $s - c = \frac{84}{10.5} = 8$. So the sides are 15, 14, and 13 units, and the area is 84 square units.

3. Find the value of the parameter p for which the following equation has a real root.

$$\sqrt{x^2 - 8x - 65} + \sqrt{x^2 - 169} + \sqrt{x^2 + (p + 14)x + 5p + 9} = 0$$

Solution: A sum of three nonnegative numbers is zero only if all three are zero. $x^2 - 8x - 65 = (x - 13)(x + 5)$ is zero for $x = 13$ or $x = -5$, and $x^2 - 169$ is zero for $x = \pm 13$. So the only possible root is $x = 13$, and then the third term is zero if

$$169 + 13(p + 14) + 5p + 9 = 0, \quad 18p + 360 = 0, \quad p = -20$$

4. The second coordinates of all three vertices of the acute triangle ABC are positive, and one side is parallel to the x -axis. The equation of the circumscribed circle is $x^2 - 8x + y^2 - 8y - 8 = 0$, and the orthocenter is $(2, 6)$. Find the radius of the inscribed circle.

Solution: The circumscribed circle is $(x - 4)^2 + (y - 4)^2 = 40$. Let BC be the side parallel to the x -axis; then the altitude from A is vertical and goes through the orthocenter $H(2, 6)$, so it is the line $x = 2$. It meets the circle at $(2, 10)$ and $(2, -2)$; A has a positive second coordinate, so $A = (2, 10)$. The reflection of the orthocenter in a side is on the circumscribed circle, so the reflection of H in BC is $(2, -2)$, and the line BC is halfway between: $y = 2$. Then $(x - 4)^2 + 4 = 40$ gives $B(-2, 2)$ and $C(10, 2)$. (Check: $\overrightarrow{BH} = (4, 4)$ is perpendicular to $\overrightarrow{AC} = (8, -8)$.)

The sides are $a = BC = 12$, $b = CA = 8\sqrt{2}$, $c = AB = 4\sqrt{5}$, and the area is $t = \frac{12 \cdot 8}{2} = 48$. From $t = \rho s$, the radius of the inscribed circle is

$$\rho = \frac{2t}{a + b + c} = \frac{96}{12 + 8\sqrt{2} + 4\sqrt{5}} = \frac{24}{3 + 2\sqrt{2} + \sqrt{5}} \approx 2.98$$

5. Solve the following equation over the real numbers.

$$\log_3^2(x^2 - 8x + 15) - [\log_3(x^2 - 8x + 15) - 1] \cdot \log_3 5 = \log_3(x^2 - 8x + 15) + [\log_3(x^2 - 8x + 15) - 1] \cdot \log_3 7$$

(Here $\log_3^2 u$ means $(\log_3 u)^2$.)

Solution: The domain is given by $x^2 - 8x + 15 = (x - 3)(x - 5) > 0$: $x < 3$ or $x > 5$. Let $y = \log_3(x^2 - 8x + 15)$ and $c = \log_3 5 + \log_3 7 = \log_3 35$. The equation is

$$y^2 - (1 + c)y + c = 0, \quad (y - 1)(y - c) = 0$$

If $y = 1$: $x^2 - 8x + 15 = 3$, $x^2 - 8x + 12 = 0$, so $x = 2$ or $x = 6$. If $y = \log_3 35$: $x^2 - 8x + 15 = 35$, $x^2 - 8x - 20 = 0$, so $x = -2$ or $x = 10$. All four are in the domain, so the solutions are $x = -2, 2, 6, 10$.

6. Find the pairs of real numbers for which

$$\sin^8 x + \sin^6 x + \sin^4 x + \sin^2 x + 1 = \sin^4 x (2^{1+\cos y} + 1)$$

Solution: Let $z = \sin^2 x$. If $z = 0$, the left-hand side is 1 and the right-hand side is 0, so $z > 0$. Dividing by z^2 and subtracting 1:

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) = 2^{1+\cos y}$$

The sum of a positive number and its reciprocal is at least 2, so the left-hand side is at least 4, with equality only for $z = 1$. The right-hand side is at most $2^2 = 4$, with equality only for $\cos y = 1$. So $\sin^2 x = 1$ and $\cos y = 1$:

$$x = \frac{\pi}{2} + k\pi, \quad y = 2m\pi, \quad k, m \text{ are integers}$$

7. On the circle with center at the origin and radius 1, find the points whose coordinates give the largest value of the expression $x^2 + xy + y^2$.

Solution: The points of the circle are $(\cos \alpha, \sin \alpha)$, and

$$x^2 + xy + y^2 = \cos^2 \alpha + \cos \alpha \sin \alpha + \sin^2 \alpha = 1 + \frac{1}{2} \sin 2\alpha$$

This is largest, $\frac{3}{2}$, when $\sin 2\alpha = 1$, that is, $\alpha = 45^\circ$ or 225° . The points are

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

(The smallest value, $\frac{1}{2}$, is at $\alpha = 135^\circ$ and 315° .)