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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. For which real numbers x is it true that

a) $(x+1)\sqrt{4-x^2} \leq 0$

b) $\cos x \geq 2 \cos \frac{x}{2} + \cot^2 \frac{\pi}{6}$

c) $(3^{x-1})^x \leq \sqrt{5^{\sin(1993\pi)}} - \log_{1/\sqrt{2}} 16$

Solution: a) We need $4-x^2 \geq 0$, that is, $-2 \leq x \leq 2$. The square root is never negative, and it is 0 for $x = \pm 2$; the product is 0 also for $x = -1$, and negative where $x+1 < 0$. The solution is $-2 \leq x \leq -1$ or $x = 2$.

b) $\cot^2 \frac{\pi}{6} = 3$ and $\cos x = 2 \cos^2 \frac{x}{2} - 1$. With $c = \cos \frac{x}{2}$, the inequality is $2c^2 - 1 \geq 2c + 3$, that is, $c^2 - c - 2 = (c-2)(c+1) \geq 0$, so $c \leq -1$ or $c \geq 2$. Since $-1 \leq c \leq 1$, only $c = -1$ is possible: $\frac{x}{2} = \pi + 2k\pi$, that is,

$$x = (4k+2)\pi, \quad k \text{ is an integer}$$

(Check: for $x = 2\pi$, $\cos x = 1$ and $2 \cos \pi + 3 = 1$.)

c) $\sin(1993\pi) = 0$, so $\sqrt{5^0} = 1$, and $\log_{1/\sqrt{2}} 16 = \frac{4}{-\frac{1}{2}} = -8$. So the right-hand side is 9, and the inequality is $3^{x^2-x} \leq 3^2$. Since 3^t is increasing, $x^2 - x - 2 = (x-2)(x+1) \leq 0$: the solution is $-1 \leq x \leq 2$.

Note: The solution sketch in KoMaL has two slips here: in b) it gives $x = (2k+1)\pi$, but for example $x = \pi$ does not satisfy the inequality; and in c) it solves the inequality with $\geq$ instead of $\leq$ (getting $x \leq -1$ or $x \geq 2$).

2. From a point outside a circle of radius r , we draw two secants to the circle: one through the center of the circle, and the other one at distance $\frac{r}{2}$ from the center. Find the perimeter and the area of the part of the disk bounded by the chords of the two secants and the arcs between them.

Solution: Let the first secant meet the circle at A and B (so AB is a diameter, with center O), and the second one at C and D , with C closer to the outside point, on the same side as A . The distance of O from CD is $\frac{r}{2}$, so in the isosceles triangle OCD the angles at C and D are 30° , the angle COD is 120° , and $CD = r\sqrt{3}$. So the central angles of the arcs AC and BD add up to $180^\circ - 120^\circ = 60^\circ$, and these arcs have total length $\frac{2r\pi}{6} = \frac{r\pi}{3}$. The region consists of the triangle OCD and the two circular sectors OAC and OBD (total central angle 60°):

$$k = 2r + r\sqrt{3} + \frac{r\pi}{3}, \quad t = \frac{r^2\sqrt{3}}{4} + \frac{r^2\pi}{6}$$

3. For a non-constant arithmetic sequence, the product of the first and second terms, the product of the second and third terms, and the product of the third and first terms are, in this order, three consecutive terms of a geometric sequence. What is the ratio of the geometric sequence?

Solution: Let the terms be $x - d$, x , $x + d$ with $d \neq 0$. The products are $g_1 = (x - d)x$, $g_2 = x(x + d)$, $g_3 = (x + d)(x - d)$, and they form a geometric sequence if $g_2^2 = g_1g_3$:

$$x^2(x + d)^2 = (x - d)^2x(x + d)$$

For a geometric sequence the terms cannot be 0, so $x \neq 0$ and $x + d \neq 0$, and we can divide: $x(x + d) = (x - d)^2$, that is, $3xd = d^2$, $d = 3x$. Then the terms are $-2x$, x , $4x$, the products are $-2x^2$, $4x^2$, $-8x^2$, and the ratio is $q = -2$.

4. About a point P in the plane of the equilateral triangle ABC we know that $PA = PB = 2\sqrt{13}$ and $PC = 2\sqrt{3}$. What is the area of the triangle?

Solution: Since $PA = PB$, P is on the perpendicular bisector of AB , which is the line through C and the midpoint F of AB . Let the side be $2x$; then $AF = x$ and $CF = x\sqrt{3}$, and $FP^2 = 52 - x^2$.

If P is beyond C (on the ray FC past C): $FP = x\sqrt{3} + 2\sqrt{3}$, so $x^2 + 3(x + 2)^2 = 52$, $x^2 + 3x - 10 = 0$, $x = 2$. The area is $x^2\sqrt{3} = 4\sqrt{3}$.

If P is between F and C : $FP = x\sqrt{3} - 2\sqrt{3}$, so $x^2 + 3(x - 2)^2 = 52$, $x^2 - 3x - 10 = 0$, $x = 5$. The area is $25\sqrt{3}$.

If P is on the other side of AB (or on AB): then $FP = PC - FC = 2\sqrt{3} - x\sqrt{3}$, which needs $x \leq 2$, and $x^2 + 3(2 - x)^2 = 52$ gives $x^2 - 3x - 10 = 0$, $x = 5$ or $x = -2$; neither is allowed.

So there are two such triangles, with areas $4\sqrt{3} \approx 6.93$ and $25\sqrt{3} \approx 43.30$.

5. Two sides of a symmetric tangential quadrilateral (a quadrilateral with an axis of symmetry and an inscribed circle) are on the lines $3x - 4y + 24 = 0$ and $y = \frac{3}{4}x - 4$, and two of its vertices are on the y -axis. What kind of quadrilateral is it, and what are the coordinates of its other vertices in the first quadrant?

Solution: The first line is $y = \frac{3}{4}x + 6$, so the two lines are parallel, and the quadrilateral is a trapezoid (possibly a rhombus). The vertices on the y -axis are $D(0, 6)$ and $A(0, -4)$, so $AD = 10$ is a side. The distance between the parallel lines is $\frac{10}{5/4} = 8$, so the radius of the inscribed circle is 4. Let $\mathbf{u} = \frac{1}{5}(4, 3)$ be the unit vector along the lines; $D - A = (0, 10)$ has component 6 along $\mathbf{u}$.

A symmetric quadrilateral with two parallel sides is a rhombus or an isosceles trapezoid.

Rhombus: all sides are 10, so the other vertices are $A + 10\mathbf{u} = (8, 2)$ and $D + 10\mathbf{u} = (8, 12)$.

Isosceles trapezoid: both legs are 10 and the height is 8, so each leg has projection 6 onto the bases, and the bases differ by 12; since the quadrilateral is tangential, the bases add up to $10 + 10 = 20$, so they are 16 and 4. The other vertices are $A + 16\mathbf{u} = \left(\frac{64}{5}, \frac{28}{5}\right)$ and $D + 4\mathbf{u} = \left(\frac{16}{5}, \frac{42}{5}\right)$.

So the quadrilateral is a rhombus with the other vertices $(8, 2)$ and $(8, 12)$, or an isosceles trapezoid with the other vertices $\left(\frac{64}{5}, \frac{28}{5}\right)$ and $\left(\frac{16}{5}, \frac{42}{5}\right)$.

6. For which values of the real parameter p are the following equations equivalent?

$$(1) \quad \sin 2x + p = \sin x + 2p \cos x$$

$$(2) \quad 2 \cos 2x + p^2 = 5p \cos x - 2$$

Solution: Using $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 2 \cos^2 x - 1$, the equations can be written as

$$(1) \quad (2 \cos x - 1)(\sin x - p) = 0, \quad (2) \quad (4 \cos x - p)(\cos x - p) = 0$$

Equation (1) is satisfied by every x with $\cos x = \frac{1}{2}$, so (2) must be too: $(2 - p)\left(\frac{1}{2} - p\right) = 0$, that is, $p = 2$ or $p = \frac{1}{2}$.

If $p = 2$: (1) means $\cos x = \frac{1}{2}$ (since $\sin x = 2$ is impossible), and (2) means $\cos x = \frac{1}{2}$ (since $\cos x = 2$ is impossible). The equations are equivalent.

If $p = \frac{1}{2}$: $x = \frac{\pi}{6}$ satisfies (1) ($\sin x = \frac{1}{2}$), but not (2), since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ is neither $\frac{1}{8}$ nor $\frac{1}{2}$.

So the equations are equivalent only for $p = 2$.

7. At most what fraction of the volume of a frustum of a right circular cone can be filled by the sphere inscribed in the frustum (the sphere that touches both bases and the lateral surface)?

Solution: Let R and r be the radii of the bases and ρ the radius of the sphere; the height is 2ρ . In the cross-section through the axis, the slant height is $R + r$ (tangent segments), and the right triangle cut off by a height gives $(2\rho)^2 + (R - r)^2 = (R + r)^2$, so $\rho^2 = rR$. Then

$$\frac{V_{\text{sphere}}}{V_{\text{frustum}}} = \frac{\frac{4}{3}\rho^3\pi}{\frac{2\rho\pi}{3}(r^2 + rR + R^2)} = \frac{2\rho^2}{r^2 + rR + R^2} = \frac{2rR}{r^2 + rR + R^2} = \frac{2}{\frac{r}{R} + 1 + \frac{R}{r}}$$

The sum of the positive number $\frac{r}{R}$ and its reciprocal is at least 2, so the ratio is at most $\frac{2}{3}$, with equality only for $r = R$, when the frustum is a cylinder. For a true frustum ($r \neq R$) the ratio is less than $\frac{2}{3}$, but it can be as close to $\frac{2}{3}$ as we like.

8. For which pairs of real numbers (x, y) is it true that

$$5x^2 + 8xy + 4y^2 = 4 + 4x \quad \text{and} \quad \cos^2(\pi x) - \sin^2(\pi y) = 1?$$

Solution: The second equation is $-\sin^2(\pi x) = \sin^2(\pi y)$; the left-hand side is not positive and the right-hand side is not negative, so both are 0: x and y are integers. The first equation can be written as

$$4(x + y)^2 + (x - 2)^2 = 8$$

With integers, the only possibility is $4(x + y)^2 = 4$ and $(x - 2)^2 = 4$: $x = 0$ or $x = 4$, and $x + y = \pm 1$. The solutions are

$$(0, 1), \quad (0, -1), \quad (4, -3), \quad (4, -5)$$