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Pdf file: KoMaL, December 1993, page 499

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The midline of the isosceles trapezoid $ABCD$ is 8 units long, and its height is 4 units.

- a) Find the area of the trapezoid.

- b) Find the perimeter of the trapezoid and the radius of its circumscribed circle, if the diagonal AC is perpendicular to the leg BC .

Solution: a) The area is the midline times the height: $8 \cdot 4 = 32$ square units.

b) Let AB be the longer base, and let C' be the projection of C onto AB , with $C'B = x$. In an isosceles trapezoid AC' is equal to the midline, 8. Since $\angle ACB = 90^\circ$, AB is a diameter of the circumscribed circle, and in the right triangle ABC the altitude theorem gives $4^2 = 8x$, so $x = 2$. Then $AB = 10$, the radius of the circumscribed circle is 5, the shorter base is $CD = 10 - 2 \cdot 2 = 6$, and by the leg theorem $BC^2 = x \cdot AB = 20$, so $BC = 2\sqrt{5}$. The perimeter is $10 + 6 + 2 \cdot 2\sqrt{5} = 16 + 4\sqrt{5}$ units.

2. For what values of the parameter k does the quadratic equation

$$x^2 - 2(3k + 2)x + k^2 + 11k - 102 = 0$$

have two real roots with opposite signs?

Solution: The roots have opposite signs exactly when their product is negative (then the discriminant is automatically positive):

$$k^2 + 11k - 102 = (k + 17)(k - 6) < 0, \quad -17 < k < 6$$

Note: In KoMaL, this problem asks for "two different real roots". That is most likely a misprint (the words "with opposite signs" are missing): the discriminant is $4(8k^2 + k + 106) > 0$ for every k , so the equation always has two different real roots, and the published solution answers the question about opposite signs.

3. Solve the following equation on the interval $0 \leq x \leq 2\pi$.

$$\sin 2x + |\sin x| + \sin \frac{11\pi}{2} = \cos 2x - 2 \cos^2 x$$

Solution: $\sin \frac{11\pi}{2} = -1$ and $\cos 2x - 2 \cos^2 x = -1$, so the equation is $\sin 2x + |\sin x| = 0$.

If $0 \leq x \leq \pi$: $\sin x \geq 0$, and $2 \sin x \cos x + \sin x = \sin x (2 \cos x + 1) = 0$, so $x = 0$, $x = \pi$, or $x = \frac{2\pi}{3}$.

If $\pi < x \leq 2\pi$: $\sin x \leq 0$, and $\sin x (2 \cos x - 1) = 0$, so $x = 2\pi$ or $x = \frac{5\pi}{3}$.

The solutions are $0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

4. For which real numbers is the following inequality true?

$$\log_{\frac{1}{2}} \sqrt{x+2} < 1 + \log_{\frac{1}{2}} \sqrt{9-x^2}$$

Solution: We need $x+2 > 0$ and $9-x^2 > 0$, that is, $-2 < x < 3$. Since $1 = \log_{\frac{1}{2}} \frac{1}{2}$, the inequality is $\log_{\frac{1}{2}} \sqrt{x+2} < \log_{\frac{1}{2}} \left(\frac{1}{2} \sqrt{9-x^2} \right)$, and since $\log_{\frac{1}{2}}$ is decreasing,

$$\sqrt{x+2} > \frac{1}{2} \sqrt{9-x^2}, \quad 4(x+2) > 9-x^2, \quad x^2+4x-1 > 0$$

(both sides are nonnegative, so squaring is allowed). So $x < -2 - \sqrt{5}$ or $x > \sqrt{5} - 2$, and with the domain the solution is $\sqrt{5} - 2 < x < 3$.

5. Find the values of the variables for which the following expressions take their largest values, and find these largest values.

$$\text{a) } \frac{16}{2^{x+2} + 2^{-x}} \quad \text{b) } -x^4 + 8x^2 - 12 - (1+x+\log_2 y)^2$$

Solution: a) The numerator is a positive constant and the denominator is positive, so the fraction is largest where the denominator is smallest. By the inequality between the arithmetic and geometric means,

$$2^{x+2} + 2^{-x} \geq 2\sqrt{2^{x+2} \cdot 2^{-x}} = 4$$

with equality if $2^{x+2} = 2^{-x}$, that is, $x = -1$. The largest value is $\frac{16}{4} = 4$, at $x = -1$.

b) The expression is $4 - (x^2 - 4)^2 - (1+x+\log_2 y)^2 \leq 4$, with equality if $x^2 = 4$ and $\log_2 y = -1-x$. The largest value is 4, at $(x, y) = \left(2, \frac{1}{8}\right)$ and at $(x, y) = (-2, 2)$.

Note: The solution sketch in KoMaL works with $(1+x-\log_2 y)^2$ and gets the points $(2, 8)$ and $\left(-2, \frac{1}{2}\right)$.

6. Prove that if the angles α, β, γ of a triangle satisfy $\frac{\sin \gamma}{\cos \beta} + \frac{\sin \beta}{\cos \gamma} = 2 \sin \alpha$, then the triangle is a right triangle.

Solution: The fractions are defined, so $\beta \neq 90^\circ$ and $\gamma \neq 90^\circ$. Multiplying by $\cos \beta \cos \gamma$ and using $\sin \alpha = \sin(\beta + \gamma)$:

$$\sin \gamma \cos \gamma + \sin \beta \cos \beta = 2 \sin(\beta + \gamma) \cos \beta \cos \gamma$$

The left-hand side is $\frac{\sin 2\gamma + \sin 2\beta}{2} = \sin(\beta + \gamma) \cos(\beta - \gamma)$. Since $\sin(\beta + \gamma) = \sin \alpha > 0$, we can divide:

$$\cos(\beta - \gamma) = 2 \cos \beta \cos \gamma, \quad \cos \beta \cos \gamma + \sin \beta \sin \gamma = 2 \cos \beta \cos \gamma, \quad \cos(\beta + \gamma) = 0$$

So $\beta + \gamma = 90^\circ$, that is, $\alpha = 90^\circ$: the triangle is a right triangle.

7. The circle k is tangent to the x -axis and to the line e with equation $3x - 4y + 45 = 0$; the second coordinate of the common point of k and e is 9. Find the equation of the circle.

Solution: The point of tangency on e is $E(-3, 9)$ (from $3x - 36 + 45 = 0$). The center is on the line through E perpendicular to e : $4x + 3y = 15$. The line e meets the x -axis at $G(-15, 0)$, and $GE = \sqrt{144 + 81} = 15$. The two tangent segments from G are equal, so the circle touches the x -axis 15 units from G : at $(-30, 0)$ or at $(0, 0)$. The center is above this point, on the line $4x + 3y = 15$: $(-30, 45)$ or $(0, 5)$. There are two such circles:

$$(x + 30)^2 + (y - 45)^2 = 45^2 \quad \text{and} \quad x^2 + (y - 5)^2 = 25$$

8. Prove that if n is a positive integer, then

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 5 + \cdots + n(n+1)(2n+1) = \frac{1}{2}n(n+1)^2(n+2)$$

Solution: By induction on n . For $n = 1$ both sides are 6. If the statement holds for n , then

$$\begin{aligned} \frac{1}{2}n(n+1)^2(n+2) + (n+1)(n+2)(2n+3) &= \frac{1}{2}(n+1)(n+2)[n(n+1) + 2(2n+3)] \\ &= \frac{1}{2}(n+1)(n+2)^2(n+3) \end{aligned}$$

since $n^2 + 5n + 6 = (n+2)(n+3)$, which is the statement for $n+1$.

(Another way: $k(k+1)(2k+1) = 2k^3 + 3k^2 + k$, and the formulas for the sums of the first n cubes, squares, and positive integers give the result.)