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translated from: KoMaL, November 1993, page 381

Pdf file: KoMaL, November 1993, page 381

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

- Two vertices of an isosceles triangle are the points $(1, 6)$ and $(-5, 2)$. What relation holds between the coordinates of the points that can be the third vertex?

Solution: Let $P(1, 6)$, $Q(-5, 2)$, and $PQ = \sqrt{36 + 16} = \sqrt{52}$.

- If PQ is the base, the third vertex is on the perpendicular bisector of PQ , but not at the midpoint $(-2, 4)$:

$$3x + 2y = 2, \quad (x, y) \neq (-2, 4)$$

- If PQ is a leg, the third vertex is $\sqrt{52}$ away from P or from Q , but not on the line PQ :

$$\begin{aligned} (x-1)^2 + (y-6)^2 &= 52, & (x, y) &\neq (-5, 2), (7, 10), & \text{ or} \\ (x+5)^2 + (y-2)^2 &= 52, & (x, y) &\neq (1, 6), (-11, -2) \end{aligned}$$

- The third term of a geometric sequence of positive integers is 6 more than the first term. What is the sum of the first three terms of the sequence?

Solution: Let the terms be a, aq, aq^2 (positive integers). Then $aq^2 = a + 6$, so $q > 1$, and $(aq)^2 = a \cdot aq^2 = a^2 + 6a$. So $a^2 + 6a$ is a perfect square, and $a^2 < a^2 + 6a < (a+3)^2$, so $a^2 + 6a = (a+1)^2$ or $(a+2)^2$. The first gives $4a = 1$, which is impossible; the second gives $2a = 4$, so $a = 2$. Then $aq = 4$, $q = 2$, the terms are 2, 4, 8, and their sum is 14.

- An isosceles triangle whose legs make a 120° angle with each other is circumscribed around a circle of radius r . Find the exact values of the perimeter and the area of the triangle.

Solution: Let C be the apex, C' the midpoint of the base AB , and O the center of the circle. The base angles are 30° . CO bisects the 120° angle, and the radius to the point of tangency on CB is perpendicular to CB , so $CO = \frac{r}{\sin 60^\circ} = \frac{2r}{\sqrt{3}}$. The height is $CC' = r + \frac{2r}{\sqrt{3}}$, and in the right triangle $CC'B$ (with a 30° angle at B),

$$C'B = \sqrt{3}CC' = (\sqrt{3} + 2)r, \quad CB = 2CC' = \left(2 + \frac{4}{\sqrt{3}}\right)r$$

So the perimeter and the area are

$$k = 2\left(\sqrt{3} + 2\right)r + 2\left(2 + \frac{4}{\sqrt{3}}\right)r = \left(8 + \frac{14\sqrt{3}}{3}\right)r, \quad t = C'B \cdot CC' = \left(4 + \frac{7\sqrt{3}}{3}\right)r^2$$

(Check: $t = r \cdot \frac{k}{2}$.)

4. Which pairs of real numbers (x, y) satisfy the following system of equations?

$$\begin{aligned}\log_x y + \log_y x &= \frac{5}{2} \\ 4\sqrt{x} - 3\sqrt{y} &= 1\end{aligned}$$

Solution: We need x and y positive and different from 1. Since $\log_y x = \frac{1}{\log_x y}$, the first equation gives $\log_x y = 2$ or $\log_x y = \frac{1}{2}$, that is, $y = x^2$ or $x = y^2$.

If $y = x^2$: $4\sqrt{x} - 3x = 1$, that is, $(3\sqrt{x} - 1)(\sqrt{x} - 1) = 0$. $x = 1$ is not allowed, so $x = \frac{1}{9}$ and $y = \frac{1}{81}$.

If $x = y^2$: $4y - 3\sqrt{y} = 1$, that is, $(\sqrt{y} - 1)(4\sqrt{y} + 1) = 0$, so $y = 1$, which is not allowed.

The only solution is $(x, y) = \left(\frac{1}{9}, \frac{1}{81}\right)$.

5. In the plane of the isosceles right triangle ABC (with the right angle at C), the point P is chosen so that $AP = BP = \sqrt{5}$ and $CP = 1$. What is the area of the triangle ABC ?

Solution: Since $AP = BP$, P is on the perpendicular bisector of AB , the line CF , where F is the midpoint of AB . Let $AF = FC = x$; the area of the triangle is x^2 .

If P is between F and C : $FP = x - 1$, and $x^2 + (x - 1)^2 = 5$ gives $x = 2$: the area is 4.

If P is beyond C : $FP = x + 1$, and $x^2 + (x + 1)^2 = 5$ gives $x = 1$: the area is 1.

If P is on the other side of AB (or on AB): $FP = 1 - x$ with $x \leq 1$, and $x^2 + (1 - x)^2 = 5$ gives $x = 2$ or $x = -1$, neither allowed.

So the area is 4 or 1 square units.

6. Find the largest subset of the real numbers on which the following expression is defined.

$$\frac{\sqrt{3 \cos x - \cos 2x - 2}}{\log(2 - x - x^2)}$$

Solution: We need three conditions.

- $3 \cos x - \cos 2x - 2 = -2 \cos^2 x + 3 \cos x - 1 = -(2 \cos x - 1)(\cos x - 1) \geq 0$, that is, $\frac{1}{2} \leq \cos x \leq 1$.
- $2 - x - x^2 = -(x + 2)(x - 1) > 0$, that is, $-2 < x < 1$.
- $2 - x - x^2 \neq 1$, that is, $x \neq \frac{-1 \pm \sqrt{5}}{2}$.

On the interval $(-2, 1)$, $\cos x \geq \frac{1}{2}$ means $x \geq -\frac{\pi}{3}$ (since $\frac{\pi}{3} > 1$), and $\frac{-1 - \sqrt{5}}{2} < -\frac{\pi}{3}$. So the expression is defined for

$$-\frac{\pi}{3} \leq x < 1, \quad x \neq \frac{\sqrt{5} - 1}{2}$$

7. We cut the cube $ABCD A' B' C' D'$ with the plane through the midpoint of the edge BB' and the vertices A and D' . What is the ratio of the volumes of the two parts of the cube?

Solution: Let the edge be 1, and let F be the midpoint of BB' . The plane meets the line $A'B'$ at M (the lines AF and $A'B'$ are in the face $ABB'A'$); the triangles ABF and $MB'F$ are congruent, so $B'M = AB = 1$ and $A'M = 2$. The line MD' meets the edge $B'C'$ at its midpoint G . So the plane cuts the cube in the trapezoid $AFGD'$, and one part is the frustum $AD'A'FG B'$.

This frustum is the pyramid $AA'D'M$ (with apex A and base the right triangle $A'D'M$ with legs 1 and 2) minus the pyramid $FB'GM$, which is similar to it with ratio $\frac{1}{2}$, so it is $\frac{7}{8}$ of the big pyramid:

$$V_1 = \frac{7}{8} \cdot \frac{1}{3} \cdot \frac{1 \cdot 2}{2} \cdot 1 = \frac{7}{24}, \quad V_2 = 1 - \frac{7}{24} = \frac{17}{24}$$

The ratio of the volumes is 7 : 17.

8. The diagonals of the quadrilateral $ABCD$ inscribed in a circle of radius 1 intersect at M . What can we say about the area of the quadrilateral, if we know that the product of the areas of the triangles ABM and CDM is $\frac{1}{4}$?

Solution: Let t_1, t_2, t_3, t_4 be the areas of the triangles ABM, BCM, CDM, DAM , and φ the angle between the diagonals. Then

$$t_1 t_3 = \frac{1}{2} MA \cdot MB \sin \varphi \cdot \frac{1}{2} MC \cdot MD \sin \varphi = \frac{1}{2} MB \cdot MC \sin \varphi \cdot \frac{1}{2} MD \cdot MA \sin \varphi = t_2 t_4$$

(the angles at M are φ and $180^\circ - \varphi$, with the same sine). So $t_1 t_3 = t_2 t_4 = \frac{1}{4}$, and by the inequality between the arithmetic and geometric means, the area is

$$T = (t_1 + t_3) + (t_2 + t_4) \geq 2\sqrt{t_1 t_3} + 2\sqrt{t_2 t_4} = 2$$

On the other hand, a quadrilateral inscribed in a circle of radius 1 has area at most 2: its area is $\frac{1}{2} AC \cdot BD \sin \varphi \leq \frac{1}{2} \cdot 2 \cdot 2 = 2$, since the diagonals are chords, at most 2 long. So the area of the quadrilateral is exactly 2 (and then both diagonals are diameters and perpendicular, so the quadrilateral is a square).