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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. One leg of a right triangle is 6.5 units long, and the altitude to the hypotenuse is 6 units. Find the other leg and the hypotenuse.

Solution: Let p and q be the projections of the given leg $a = 6.5$ and of the other leg b onto the hypotenuse c . By the Pythagorean theorem $p^2 = 6.5^2 - 6^2 = 6.25$, so $p = 2.5$. By the altitude theorem $6^2 = pq$, so $q = 14.4$, and $c = p + q = 16.9$. By the leg theorem $b^2 = qc = 14.4 \cdot 16.9 = 243.36$, so $b = 15.6$. The other leg is 15.6 units and the hypotenuse is 16.9 units long.

2. In triangle ABC , $AB = 1$ unit, $\angle BAC = 45^\circ$, and $\angle ABC = 120^\circ$. Find the exact lengths of the sides AC and BC .

Solution: The third angle is 15° , and $\sin 15^\circ = \sin (45^\circ - 30^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$. By the Law of Sines,

$$BC = \frac{\sin 45^\circ}{\sin 15^\circ} = \frac{2\sqrt{2}}{\sqrt{6} - \sqrt{2}} = \sqrt{3} + 1, \quad AC = \frac{\sin 120^\circ}{\sin 15^\circ} = \frac{2\sqrt{3}}{\sqrt{6} - \sqrt{2}} = \frac{3\sqrt{2} + \sqrt{6}}{2}$$

(that is, $BC \approx 2.73$ and $AC \approx 3.35$).

3. Of five numbers, the first four are consecutive terms of an arithmetic sequence, and the last three are consecutive terms of a geometric sequence. Find these five numbers, if the sum of the first four is -36 , and the product of the second and third is 72.

Solution: Let the first four numbers be $a - 3t$, $a - t$, $a + t$, $a + 3t$. Then $4a = -36$, so $a = -9$, and $a^2 - t^2 = 72$, so $t^2 = 9$.

If $t = -3$: the first four numbers are 0, -6 , -12 , -18 , and the fifth one is $\frac{(-18)^2}{-12} = -27$ (the geometric sequence $-12, -18, -27$ has ratio $\frac{3}{2}$).

If $t = 3$: the first four numbers are $-18, -12, -6, 0$. The last three numbers would be $-6, 0$, and a fifth number; this is a geometric sequence only if we allow the ratio 0, and then the fifth number is 0.

So the numbers are 0, -6 , -12 , -18 , -27 (and, if a geometric sequence may have ratio 0, also $-18, -12, -6, 0, 0$).

4. The coordinates of the point P are $(4, 1)$, the equation of the line e is $x - y = -1$, and the equation of the line f is $x + 2y = 11$. Find the equation of the line through P whose segment between e and f is divided by P into thirds, with P being the point of division closer to f .

Solution: Let the endpoints be $E(t, t + 1)$ on e and $F(11 - 2p, p)$ on f . P is closer to F , so $P = \frac{E + 2F}{3}$:

$$\frac{t + 2(11 - 2p)}{3} = 4, \quad \frac{t + 1 + 2p}{3} = 1$$

From the second equation $t = 2 - 2p$, and then the first one gives $24 - 6p = 12$, so $p = 2$ and $t = -2$. So $E(-2, -1)$ and $F(7, 2)$, and the line EF (which has slope $\frac{1}{3}$) is

$$x - 3y = 1$$

5. Solve the equation $\sqrt{4 \cdot \log_2 x - (\log_2 x)^2} = p$ over the real numbers, where p is a real parameter.

Solution: We need $x > 0$. Let $u = \log_2 x$; the equation is $\sqrt{4u - u^2} = p$.

If $p < 0$: there is no solution.

If $p \geq 0$: squaring, $u^2 - 4u + p^2 = 0$, so $u = 2 \pm \sqrt{4 - p^2}$ (if $p \leq 2$). So:

- $p = 0$: $u = 0$ or $u = 4$, that is, $x = 1$ or $x = 16$.
- $0 < p < 2$: two solutions, $x = 2^{2+\sqrt{4-p^2}}$ and $x = 2^{2-\sqrt{4-p^2}}$.
- $p = 2$: $u = 2$, that is, $x = 4$.
- $p > 2$: there is no solution.

6. Solve the following inequality over the real numbers.

$$\frac{x^2}{(1 - \sqrt{1 - x})^2} < 9 - x$$

Solution: We need $1 - x \geq 0$ and $\sqrt{1 - x} \neq 1$, that is, $x \leq 1$ and $x \neq 0$. Let $y = \sqrt{1 - x} \geq 0$, $y \neq 1$; then $x = 1 - y^2 = (1 - y)(1 + y)$, and the inequality is

$$\frac{(1 - y)^2 (1 + y)^2}{(1 - y)^2} < 9 - (1 - y^2), \quad (1 + y)^2 < 8 + y^2, \quad y < \frac{7}{2}$$

So $1 - x < \frac{49}{4}$, that is, $x > -\frac{45}{4}$. With the conditions, the solution is

$$-\frac{45}{4} < x < 0 \quad \text{or} \quad 0 < x \leq 1$$

7. Prove that if α, β, γ are the angles of a triangle, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma = 1$$

Solution: Since $\gamma = 180^\circ - (\alpha + \beta)$, $\cos \gamma = -\cos(\alpha + \beta)$. Also $\cos^2 \alpha + \cos^2 \beta = 1 + \frac{\cos 2\alpha + \cos 2\beta}{2} = 1 + \cos(\alpha + \beta) \cos(\alpha - \beta)$. So the left-hand side is

$$\begin{aligned} & 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) + \cos^2(\alpha + \beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\ &= 1 + \cos(\alpha + \beta) [\cos(\alpha - \beta) + \cos(\alpha + \beta) - 2 \cos \alpha \cos \beta] \end{aligned}$$

and the bracket is 0, because $\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2 \cos \alpha \cos \beta$. So the left-hand side is 1.

8. Find the integers k and n for which the roots of the equation

$$(2k - 1)nx^2 + (2k - 1)n(k - n - 4)x - 2(2k - 1)n(k - n - 2) - 1 = 0$$

are also integers.

Solution: If $n = 0$, the equation is $-1 = 0$. Otherwise ($2k - 1$ is never 0 for an integer k) the equation can be written as

$$(2k - 1)n[x^2 + (k - n - 4)x - 2(k - n - 2)] = 1$$

For an integer root x , both factors are integers with product 1, so $(2k - 1)n = \pm 1$: $k = 1$ or $k = 0$, and $n = \pm 1$.

$k = 1, n = 1$: $x^2 - 4x + 4 = 1$, that is, $x^2 - 4x + 3 = 0$, with roots 1 and 3.

$k = 1, n = -1$: $-(x^2 - 2x) = 1$, that is, $(x - 1)^2 = 0$, with the double root 1.

$k = 0, n = 1$: $-(x^2 - 5x + 6) = 1$, that is, $x^2 - 5x + 7 = 0$, which has no real root.

$k = 0, n = -1$: $x^2 - 3x + 2 = 1$, that is, $x^2 - 3x + 1 = 0$, whose roots $\frac{3 \pm \sqrt{5}}{2}$ are not integers.

So $k = 1$ and $n = \pm 1$.