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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The roots of the quadratic equation $2x^2 - 4x + 1 = 0$ are the radii of two circles. Without finding the roots, find the sum of the circumferences and the sum of the areas of the circles.

Solution: The discriminant is $16 - 8 > 0$, and the roots have positive sum and product, so they are positive. By Viète's formulas $r_1 + r_2 = 2$ and $r_1 r_2 = \frac{1}{2}$. The sum of the circumferences is $2\pi(r_1 + r_2) = 4\pi$, and the sum of the areas is

$$\pi(r_1^2 + r_2^2) = \pi((r_1 + r_2)^2 - 2r_1 r_2) = 3\pi$$

2. The sides of the rectangle $KLMN$ satisfy $KL : LM = 3 : 2$, and F is the midpoint of the side LM . A line is perpendicular to the segment KF and bisects the area of $KLMN$. In what ratio does this line divide the sides KL and MN ?

Solution: A line bisects the area of a rectangle if and only if it goes through its center (the rectangle is symmetric about its center). Use coordinates: $K(0, 0)$, $L(3, 0)$, $M(3, 2)$, $N(0, 2)$; then $F(3, 1)$, the center is $O\left(\frac{3}{2}, 1\right)$, and $\overrightarrow{KF} = (3, 1)$. The line through O perpendicular to KF is $3x + y = \frac{11}{2}$. It meets KL ($y = 0$) at $A\left(\frac{11}{6}, 0\right)$ and MN ($y = 2$) at $B\left(\frac{7}{6}, 2\right)$. So

$$KA : AL = \frac{11}{6} : \frac{7}{6} = 11 : 7, \quad MB : BN = \frac{11}{6} : \frac{7}{6} = 11 : 7$$

3. Find the real numbers x for which

$$\frac{\sqrt{x+2}-1}{\sqrt{x+3}-2} : \frac{\sqrt{x+4}-3}{\sqrt{x+13}-4} \geq 0$$

Solution: The expression is defined if $x \geq -2$, $\sqrt{x+3} \neq 2$, $\sqrt{x+4} \neq 3$, $\sqrt{x+13} \neq 4$, that is, $x \geq -2$, $x \neq 1, 5, 3$.

Using $\sqrt{u} - v = \frac{u - v^2}{\sqrt{u} + v}$ in each of the four factors,

$$\sqrt{x+2}-1 = \frac{x+1}{\sqrt{x+2}+1}, \quad \sqrt{x+3}-2 = \frac{x-1}{\sqrt{x+3}+2}, \quad \sqrt{x+4}-3 = \frac{x-5}{\sqrt{x+4}+3}, \quad \sqrt{x+13}-4 = \frac{x-3}{\sqrt{x+13}+4}$$

and the denominators here are positive. So the expression has the same sign as $\frac{(x+1)(x-3)}{(x-1)(x-5)}$, and it is 0 only for $x = -1$. A sign chart gives the solution

$$-2 \leq x \leq -1, \quad \text{or} \quad 1 < x < 3, \quad \text{or} \quad x > 5$$

4. A right circular cone is cut by planes parallel to the base into a cone and two frustums. Their volumes are in the ratio $1 : 2 : 3$, starting from the apex. What is the ratio of the heights of these solids?

Solution: The cut planes cut off cones from the apex whose volumes are 1, $1 + 2 = 3$, and $1 + 2 + 3 = 6$ units (the last one is the whole cone). These cones are similar, and the volumes of similar solids are proportional to the cubes of their sizes, so their heights are proportional to $1, \sqrt[3]{3}, \sqrt[3]{6}$. The heights of the three pieces are the differences:

$$1 : (\sqrt[3]{3} - 1) : (\sqrt[3]{6} - \sqrt[3]{3}) \approx 1 : 0.442 : 0.375$$

5. The equation of the axis of symmetry of the isosceles triangle PQR is $2y - x - 6 = 0$, and the points $P(2, 4)$ and $Q(-2, -1)$ are given. Find the length of the altitude from the missing vertex.

Solution: P is on the axis of symmetry ($8 - 2 - 6 = 0$), so P is the apex, and R is the reflection of Q in the axis. The axis has normal vector $(1, -2)$, so $R = Q + t(1, -2)$, and the midpoint $\left(-2 + \frac{t}{2}, -1 - t\right)$ is on the axis:

$$2(-1 - t) - \left(-2 + \frac{t}{2}\right) - 6 = 0, \quad t = -\frac{12}{5}, \quad R\left(-\frac{22}{5}, \frac{19}{5}\right)$$

The area of the triangle is $\frac{1}{2} \left| (-4) \left(-\frac{1}{5}\right) - (-5) \left(-\frac{32}{5}\right) \right| = \frac{78}{5}$ (using $\overrightarrow{PQ} = (-4, -5)$ and $\overrightarrow{PR} = \left(-\frac{32}{5}, -\frac{1}{5}\right)$), and $PQ = \sqrt{41}$. So the altitude from R to the side PQ is

$$m = \frac{2 \cdot \frac{78}{5}}{\sqrt{41}} = \frac{156}{5\sqrt{41}} \approx 4.87$$

6. Three sides of an isosceles trapezoid have the same length. Find its angles so that its area is as large as possible.

Solution: Let the legs and the shorter base be b , and let x be the projection of a leg onto the longer base, so the longer base is $b + 2x$ and the height is $m = \sqrt{b^2 - x^2}$ ($0 < x < b$). The area is $t = (b + x)m$, and

$$3t^2 = 3(b + x)^2(b^2 - x^2) = (3b - 3x)(b + x)^3 \leq \left(\frac{(3b - 3x) + 3(b + x)}{4} \right)^4 = \left(\frac{3b}{2} \right)^4$$

by the inequality between the arithmetic and geometric means of the four positive numbers $3b - 3x, b + x, b + x, b + x$. Equality holds if $3b - 3x = b + x$, that is, $x = \frac{b}{2}$. Then the angles at the longer base are 60° (since $\cos \alpha = \frac{x}{b} = \frac{1}{2}$), and the angles at the shorter base are 120° . (The trapezoid is half of a regular hexagon.)

(If the longer base and the legs were the equal sides, the area would have no maximum: it would grow as the trapezoid approaches a square.)

7. Solve the following system of equations over the integers.

$$\begin{aligned} ab + c &= 11 \\ bc + a &= 13 \\ ca + b &= 17 \end{aligned}$$

Solution: From the third equation $b = 17 - ac$. Substituting into the other two:

$$a(17 - ac) + c = 11, \quad c(17 - ac) + a = 13$$

Subtracting the first one from the second one: $17(c - a) - ac(c - a) + (a - c) = 2$, that is, $(c - a)(16 - ac) = 2$. For integers, the possibilities are

$$(c - a, 16 - ac) = (-2, -1), (-1, -2), (1, 2), (2, 1)$$

that is, $ac = 17, 18, 14, 15$ with $c - a = -2, -1, 1, 2$. Only the last one gives integers: $c - a = 2, ac = 15$, so $(a, c) = (3, 5)$ or $(-5, -3)$. Then $b = 17 - 15 = 2$, and checking the first equation, only $a = 3, c = 5$ works ($6 + 5 = 11$; for $a = -5, c = -3$ we would get $-10 - 3 \neq 11$). The only solution is $a = 3, b = 2, c = 5$.

8. Find the smallest value of the expression $\sin^8 x + \cos^8 x$.

Solution: Let $\sin^2 x = \frac{1}{2} + y$ and $\cos^2 x = \frac{1}{2} - y$ (with $-\frac{1}{2} \leq y \leq \frac{1}{2}$). Then

$$\sin^8 x + \cos^8 x = \left(\frac{1}{2} + y \right)^4 + \left(\frac{1}{2} - y \right)^4 = \frac{1}{8} + 3y^2 + 2y^4 \geq \frac{1}{8}$$

with equality for $y = 0$, that is, $\sin^2 x = \frac{1}{2}$. The smallest value is $\frac{1}{8}$ (at $x = \frac{\pi}{4} + \frac{k\pi}{2}$).