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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned}x^2 - 2xy + y^2 &= 9 \\4x^2 + xy + 4y^2 &= 18\end{aligned}$$

Solution: The first equation is $(x - y)^2 = 9$, so $x - y = 3$ or $x - y = -3$.

If $x = y + 3$: the second equation gives $9y^2 + 27y + 18 = 0$, that is, $y^2 + 3y + 2 = 0$, so $y = -1$, $x = 2$ or $y = -2$, $x = 1$.

If $x = y - 3$: similarly $y^2 - 3y + 2 = 0$, so $y = 1$, $x = -2$ or $y = 2$, $x = -1$.

The solutions are $(2, -1)$, $(1, -2)$, $(-2, 1)$, $(-1, 2)$. (Another way: subtracting 2 times the first equation from the second one gives $2x^2 + 5xy + 2y^2 = 0$, that is, $(2x + y)(x + 2y) = 0$.)

2. In triangle ABC , $AB = 21$, $AC = 28$, $BC = 20$ units. The angle bisector from A meets the side BC at D , and the angle bisector from C meets the side AB at E . What fraction of the area of triangle ABC is the area of triangle AED ?

Solution: Let t be the area of ABC . By the angle bisector theorem $BD : DC = AB : AC = 21 : 28 = 3 : 4$, so $BD = \frac{3}{7}BC$, and the triangle ABD has area $\frac{3}{7}t$ (same height from A). Similarly $AE : EB = CA : CB = 28 : 20 = 7 : 5$, so $AE = \frac{7}{12}AB$, and the triangle AED (same height from D as ABD) has area

$$\frac{7}{12} \cdot \frac{3}{7}t = \frac{t}{4}$$

So the area of AED is one fourth of the area of ABC .

3. Take the sum of the first n positive integers that leave remainder 1 when divided by 3, and divide it by the sum of the first n positive integers that leave remainder 3 when divided by 4. What is n , if the quotient is $\frac{5}{7}$?

Solution: The first numbers are 1, 4, 7, ... (first term 1, difference 3), with sum $S_1 = \frac{n(3n-1)}{2}$; the second ones are 3, 7, 11, ... (first term 3, difference 4), with sum $S_2 = \frac{n(4n+2)}{2}$. So

$$\frac{3n-1}{4n+2} = \frac{5}{7}, \quad 21n-7 = 20n+10, \quad n = 17$$

4. For which real numbers x is the value of the expression

$$\frac{2x^2 - 6x + 6}{x^2 - 4x + 5}$$

at least 1 and at most 3?

Solution: The denominator is $(x - 2)^2 + 1 > 0$, so we can multiply by it:

$$x^2 - 4x + 5 \leq 2x^2 - 6x + 6 \leq 3x^2 - 12x + 15$$

The first inequality is $0 \leq x^2 - 2x + 1 = (x - 1)^2$, and the second one is $0 \leq x^2 - 6x + 9 = (x - 3)^2$. Both are true for every x , so the statement holds for every real number x .

5. A convex octagon is inscribed in a circle of radius r , so that four of its sides are 1 unit long, and the other four sides are 3 units long. Find the radius of the circle.

Solution: Let α and β be the central angles over the sides of length 1 and 3; then $4\alpha + 4\beta = 360^\circ$, so $\alpha + \beta = 90^\circ$. The order of the sides does not change the central angles, and somewhere a side $AB = 1$ is followed by a side $BC = 3$. The arc AC through B is 90° , so $AC = r\sqrt{2}$, and the inscribed angle ABC (over the other arc of 270°) is 135° . By the Law of Cosines,

$$2r^2 = 1 + 9 - 2 \cdot 1 \cdot 3 \cos 135^\circ = 10 + 3\sqrt{2}, \quad r = \sqrt{5 + \frac{3\sqrt{2}}{2}} \approx 2.67$$

6. Find the equation of the circle that is tangent to the coordinate axes and also to the line $3x + 4y = 10$.

Solution: Let the center be (u, v) . It is equally far from the two axes and from the line:

$$|u| = |v| = \frac{|3u + 4v - 10|}{5}$$

If $v = u$: $|7u - 10| = 5|u|$ gives $u = 5$ or $u = \frac{5}{6}$. If $v = -u$: $|-u - 10| = 5|u|$ gives $u = \frac{5}{2}$ or $u = -\frac{5}{3}$. There are four such circles:

$$\begin{aligned} (x - 5)^2 + (y - 5)^2 &= 25, & \left(x - \frac{5}{6}\right)^2 + \left(y - \frac{5}{6}\right)^2 &= \frac{25}{36}, \\ \left(x + \frac{5}{3}\right)^2 + \left(y - \frac{5}{3}\right)^2 &= \frac{25}{9}, & \left(x - \frac{5}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 &= \frac{25}{4} \end{aligned}$$

7. For which values of the real parameter m does the equation

$$\frac{m}{12 \sin^4 x - 12 \sin^2 x + m} = 3$$

have a real solution? Solve the equation for $m = \frac{9}{2}$.

Solution: If $m = 0$, the left-hand side is 0 (or undefined), so there is no solution. If $m \neq 0$, the equation means $3(12 \sin^4 x - 12 \sin^2 x + m) = m$ (then the denominator is $\frac{m}{3} \neq 0$). With $u = \sin^2 x$:

$$18u^2 - 18u + m = 0, \quad u = \frac{1}{2} \pm \frac{1}{6}\sqrt{9 - 2m}$$

which is real for $m \leq \frac{9}{2}$. We need a root with $0 \leq u \leq 1$. If $m < 0$, then $\sqrt{9 - 2m} > 3$, and both roots are outside $[0, 1]$; if $0 < m \leq \frac{9}{2}$, then $\sqrt{9 - 2m} < 3$, and both roots are in $[0, 1]$. So the equation has a solution exactly for

$$0 < m \leq \frac{9}{2}$$

For $m = \frac{9}{2}$: $\sin^2 x = \frac{1}{2}$, so $x = \frac{\pi}{4} + n \cdot \frac{\pi}{2}$ (n is an integer).

8. Prove that one root of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) is three times the other root if and only if the discriminant of the equation is $D = \frac{b^2}{4}$.

Solution: If the roots are x_1 and $x_2 = 3x_1$, then by Viète's formulas $4x_1 = -\frac{b}{a}$ and $3x_1^2 = \frac{c}{a}$. So $3 \cdot \frac{b^2}{16a^2} = \frac{c}{a}$, that is, $4ac = \frac{3}{4}b^2$, and $D = b^2 - 4ac = \frac{b^2}{4}$.

Conversely, if $D = \frac{b^2}{4}$, then $\sqrt{D} = \frac{|b|}{2}$, and the roots are

$$x_{1,2} = \frac{-b \pm \frac{b}{2}}{2a}, \quad x_1 = -\frac{b}{4a}, \quad x_2 = -\frac{3b}{4a}$$

so $x_2 = 3x_1$.