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translated from: KoMaL, March 1994, page 110

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\frac{2}{x^2 - 9} = \frac{1}{x^2 - 3x} - \frac{x - 6}{x^2 + 3x}$$

Solution: We need $x \neq 0$, $x \neq 3$, $x \neq -3$. Multiplying by $x(x-3)(x+3)$:

$$2x = (x+3) - (x-6)(x-3), \quad x^2 - 8x + 15 = 0, \quad x = 3 \text{ or } x = 5$$

$x = 3$ is not allowed, so the only solution is $x = 5$.

2. The shorter side, the longer side, and the diagonal of a rectangle are, in this order, consecutive terms of a geometric sequence. What is the ratio of the geometric sequence? Find the angles between the diagonals and the sides.

Solution: Let the sides be a and aq and the diagonal aq^2 , with $q > 1$. By the Pythagorean theorem $a^2q^4 = a^2q^2 + a^2$, that is, $q^4 - q^2 - 1 = 0$, so

$$q^2 = \frac{1 + \sqrt{5}}{2}, \quad q = \sqrt{\frac{1 + \sqrt{5}}{2}} \approx 1.272$$

The diagonal makes the angle α with the shorter side and $\beta = 90^\circ - \alpha$ with the longer side, where $\tan \alpha = q$: $\alpha \approx 51.83^\circ$ and $\beta \approx 38.17^\circ$.

3. Solve the following equations over the real numbers.

$$\text{a) } \sqrt{(\sqrt{x})^{\log x^2}} = 100 \quad \text{b) } 10^{2x+3} = 2^{x+6} \cdot 125^x$$

Solution: a) We need $x > 0$. Then $(\sqrt{x})^{\log x^2} = x^{\frac{1}{2} \cdot 2 \log x} = x^{\log x}$, so the equation is $x^{\frac{\log x}{2}} = 10^2$. Taking the logarithm: $\frac{(\log x)^2}{2} = 2$, so $\log x = \pm 2$, and the solutions are $x = 100$ and $x = 0.01$.

b) $10^{2x+3} = 2^{2x+3} \cdot 5^{2x+3}$ and $2^{x+6} \cdot 125^x = 2^{x+6} \cdot 5^{3x}$. Dividing: $2^{x-3} = 5^{x-3}$, that is, $\left(\frac{2}{5}\right)^{x-3} = 1$, so $x = 3$.

4. The sum of the squares of the second and fourth terms of an arithmetic sequence is twice the sum of the squares of the first and third terms. The sum of the first four terms of the sequence is 20. Find the first four terms.

Solution: Let the terms be $a - d, a, a + d, a + 2d$. Then $4a + 2d = 20$, and

$$a^2 + (a + 2d)^2 = 2((a - d)^2 + (a + d)^2), \quad 2a^2 + 4ad + 4d^2 = 4a^2 + 4d^2, \quad a(a - 2d) = 0$$

If $a = 0$: $d = 10$, and the terms are $-10, 0, 10, 20$. If $a = 2d$: $10d = 20$, so $d = 2, a = 4$, and the terms are $2, 4, 6, 8$.

5. Two adjacent vertices of the rectangle $ABCD$ are $A(2, 0)$ and $D(-1, 1)$, and we know that $3 \cdot AD = AB$. How long are the segments that the circle circumscribed around the rectangle cuts off from the x -axis and from the y -axis?

Solution: $\overrightarrow{AD} = (-3, 1)$, and $\overrightarrow{AB}$ is perpendicular to it and three times as long: $\overrightarrow{AB} = (3, 9)$ or $(-3, -9)$. So $B_1(5, 9)$ or $B_2(-1, -9)$. The center of the circumscribed circle is the midpoint of the diagonal BD , and the radius is $\frac{BD}{2} = \frac{\sqrt{10 + 90}}{2} = 5$.

For B_1 : the circle is $(x - 2)^2 + (y - 5)^2 = 25$. It touches the x -axis (at $(2, 0) = A$), so the segment on the x -axis has length 0; on the y -axis, $(y - 5)^2 = 21$, so the segment is $2\sqrt{21} \approx 9.17$.

For B_2 : the circle is $(x + 1)^2 + (y + 4)^2 = 25$. On the x -axis $(x + 1)^2 = 9$, so the segment is 6; on the y -axis $(y + 4)^2 = 24$, so the segment is $4\sqrt{6} \approx 9.80$.

6. Two perpendicular chords of a circle of radius 1 intersect each other so that the segments on one chord are in the ratio $2 : 3$, and on the other one in the ratio $3 : 8$. Find the lengths of the four arcs into which the chords divide the circle.

Solution: Let the chords AC and BD meet at P , with $AP = 3x, PC = 8x, DP = 2y, PB = 3y$. By the intersecting chords theorem $3x \cdot 8x = 2y \cdot 3y$, so $y = 2x$: $AP = 3x, PB = 6x, PC = 8x, PD = 4x$. The triangles APB, BPC, CPD, DPA have right angles at P , so the inscribed angles are

$$\tan \angle ACB = \frac{6}{8}, \quad \tan \angle BDC = \frac{8}{4}, \quad \tan \angle CAD = \frac{4}{3}, \quad \tan \angle DBA = \frac{3}{6}$$

In a circle of radius 1, an arc is twice the inscribed angle over it (in radians). So the arcs are

$$\begin{aligned} \widehat{AB} &= 2 \tan^{-1} \frac{3}{4} \approx 1.287, & \widehat{BC} &= 2 \tan^{-1} 2 \approx 2.214, \\ \widehat{CD} &= 2 \tan^{-1} \frac{4}{3} \approx 1.855, & \widehat{DA} &= 2 \tan^{-1} \frac{1}{2} \approx 0.927 \end{aligned}$$

(Check: their sum is 2π .)

7. We want to make a box in the shape of a rectangular box whose base has area 4 square units, and the sum of the lengths of its 12 edges is 40 units. How should we choose the dimensions of the box so that its surface area is as large as possible? What is this largest surface area?

Solution: Let the base edges be x and y , and the height z . Then $xy = 4$ and $4(x + y + z) = 40$, so $x + y = 10 - z$. The surface area is

$$A = 2(xy + (x + y)z) = 2(4 + (10 - z)z) = 58 - 2(z - 5)^2 \leq 58$$

with equality for $z = 5$. Then $x + y = 5$ and $xy = 4$, so the base edges are 1 and 4. So the box should be $1 \times 4 \times 5$, and the largest surface area is 58 square units.

8. Solve the following equation over the pairs of real numbers.

$$\log_2 \left(\cos^2(xy) + \frac{1}{\cos^2(xy)} \right) = \frac{2}{y^2 - 4y + 6}$$

Solution: We need $\cos(xy) \neq 0$. A positive number plus its reciprocal is at least 2, with equality only for 1, so the left-hand side is at least $\log_2 2 = 1$, with equality only if $\cos^2(xy) = 1$. The right-hand side is $\frac{2}{(y-2)^2 + 2} \leq 1$, with equality only if $y = 2$. So both sides must be 1: $y = 2$ and $\cos^2(2x) = 1$, that is, $2x = k\pi$. The solutions are

$$x = \frac{k\pi}{2}, \quad y = 2, \quad k \text{ is an integer}$$