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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$\frac{xy}{x+2y} + \frac{x+2y}{xy} = 2, \quad \frac{xy}{x-y} + \frac{x-y}{xy} = \frac{17}{4}$$

Solution: We need $xy \neq 0$, $x+2y \neq 0$, $x-y \neq 0$. Let $u = \frac{xy}{x+2y}$ and $v = \frac{xy}{x-y}$. Then $u + \frac{1}{u} = 2$ gives $u = 1$, and $v + \frac{1}{v} = \frac{17}{4}$ gives $v = 4$ or $v = \frac{1}{4}$. So $xy = x+2y$, and

- $xy = 4(x-y)$: then $x+2y = 4x-4y$, so $x = 2y$, and $2y^2 = 4y$ gives $y = 2$, $x = 4$;
- $4xy = x-y$: then $4(x+2y) = x-y$, so $x = -3y$, and $-3y^2 = -y$ gives $y = \frac{1}{3}$, $x = -1$.

The solutions are $(4, 2)$ and $\left(-1, \frac{1}{3}\right)$.

2. The radii of two intersecting circles are 17 and 39 units, and the distance between their centers is 44 units. Find the length of the common chord.

Solution: The two centers and one common point form a triangle with sides 17, 39, 44, and the altitude of this triangle to the side 44 is half of the common chord (the common chord is perpendicular to the line of the centers and is bisected by it). By Heron's formula with $s = 50$,

$$t = \sqrt{50 \cdot 33 \cdot 11 \cdot 6} = 330$$

so the altitude is $\frac{2 \cdot 330}{44} = 15$, and the common chord is 30 units long.

3. We got a loan of 12 000 Ft for a purchase, at 2% monthly compound interest. At the end of every month we pay back the same amount. What is this amount, if we pay the last installment at the end of the 12th month?

Solution: Let the monthly payment be t and $q = 1.02$. At the end of the 12th month the loan has grown to $12\,000q^{12}$, and the payments, with their interest, are worth $t(q^{11} + q^{10} + \dots + 1) = t \cdot \frac{q^{12} - 1}{q - 1}$. These are equal, so

$$t = \frac{12\,000 \cdot 1.02^{12} \cdot 0.02}{1.02^{12} - 1} \approx 1134.72$$

The monthly payment is about 1135 Ft.

4. a) Which pairs of real numbers (a, b) satisfy $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$?
 b) Solve the following equation over the real numbers.

$$\sqrt{x^2 - 2x + 2} = \sqrt{x^2 - 4} + \sqrt{6 - 2x}$$

Solution: a) We need $a \geq 0$ and $b \geq 0$. Then squaring is an equivalent step: $a + b = a + b + 2\sqrt{ab}$, that is, $ab = 0$. The pairs are $a = 0, b \geq 0$ and $a \geq 0, b = 0$.

b) Since $(x^2 - 4) + (6 - 2x) = x^2 - 2x + 2$, by part a) the equation holds exactly when $x^2 - 4 \geq 0, 6 - 2x \geq 0$, and one of them is 0. $x^2 - 4 = 0$ gives $x = \pm 2$ (and then $6 - 2x > 0$), and $6 - 2x = 0$ gives $x = 3$ (and then $x^2 - 4 = 5 > 0$). The solutions are $x = -2, 2, 3$.

5. a) Factor the expression $a^3 - 3a^2b - ab^2 + 3b^3$.
 b) Solve the following equations.

$$(1) \quad 27^x - 3 \cdot 18^x - 12^x + 3 \cdot 8^x = 0$$

$$(2) \quad \sin^3 x - 3 \sin^2 x \cdot \cos x - \sin x \cdot \cos^2 x + 3 \cos^3 x = 0$$

Solution: a) $a^3 - 3a^2b - ab^2 + 3b^3 = a^2(a - 3b) - b^2(a - 3b) = (a - b)(a + b)(a - 3b)$.

b) (1) With $a = 3^x$ and $b = 2^x$, the equation is exactly the expression of part a): $(3^x - 2^x)(3^x + 2^x)(3^x - 3 \cdot 2^x) = 0$. The middle factor is positive, so $3^x = 2^x$, that is, $x = 0$, or $\left(\frac{3}{2}\right)^x = 3$, that is, $x = \log_{\frac{3}{2}} 3 \approx 2.71$.

(2) With $a = \sin x$ and $b = \cos x$: $(\sin x - \cos x)(\sin x + \cos x)(\sin x - 3 \cos x) = 0$. ($\cos x = 0$ is not a solution, since then the left-hand side is $\sin^3 x = \pm 1$.) So $\tan x = 1$, $\tan x = -1$, or $\tan x = 3$:

$$x = 45^\circ + k \cdot 180^\circ, \quad x = 135^\circ + k \cdot 180^\circ, \quad x \approx 71.57^\circ + k \cdot 180^\circ$$

6. Find the equation of the circle that is tangent to the x -axis and also to the lines $2x - y = 6$ and $2x - y = -14$.

Solution: The two lines are parallel, and their distance $\frac{|6 - (-14)|}{\sqrt{5}} = 4\sqrt{5}$ is the diameter of the circle, so $r = 2\sqrt{5}$. The center is on the line halfway between them, $2x - y = -4$, that is, $y = 2x + 4$. The circle is tangent to the x -axis, so the second coordinate of the center is $\pm 2\sqrt{5}$. There are two such circles:

$$(x + 2 - \sqrt{5})^2 + (y - 2\sqrt{5})^2 = 20 \quad \text{and} \quad (x + 2 + \sqrt{5})^2 + (y + 2\sqrt{5})^2 = 20$$

7. A convex hexagon is inscribed in a circle of radius $r = \sqrt{13}$ units, so that three of its sides are 2 units long. Find the lengths of the other three sides, if they are equal.

Solution: Let the other sides be x . The central angles over the sides of length 2 and x add up to $\frac{360^\circ}{3} = 120^\circ$, and somewhere a side $AB = 2$ is followed by a side $BC = x$. The arc AC through B is 120° , so $AC = 2r \sin 60^\circ = \sqrt{39}$, and the inscribed angle ABC is $180^\circ - 60^\circ = 120^\circ$. By the Law of Cosines,

$$39 = 4 + x^2 - 2 \cdot 2 \cdot x \cos 120^\circ, \quad x^2 + 2x - 35 = 0, \quad x = 5$$

(the other root is negative). The other three sides are 5 units long.

8. For given real numbers p and q , find the first term, the difference, and the third term of every arithmetic sequence in which the sum of the first two terms is p , and the quotient of the second and the first terms is q .

Solution: Let the first term be $a_1 \neq 0$ and the difference d . The conditions are $2a_1 + d = p$ and $a_1 + d = qa_1$.

Subtracting: $a_1(1 + q) = p$.

- If $q \neq -1$ and $p \neq 0$: $a_1 = \frac{p}{q+1}$, $d = (q-1)a_1 = \frac{p(q-1)}{q+1}$, $a_3 = a_1 + 2d = \frac{p(2q-1)}{q+1}$.
- If $q \neq -1$ and $p = 0$: $a_1 = 0$, which is not allowed: no such sequence.
- If $q = -1$ and $p \neq 0$: no such sequence.
- If $q = -1$ and $p = 0$: every $a_1 \neq 0$ works, with $d = -2a_1$ and $a_3 = -3a_1$.