

author: Imre Rábai

translated from: KoMaL, October 1994, pages 350-351

Pdf file: KoMaL, October 1994, pages 350-351

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In triangle ABC , $AB = 10$, $BC = 9$ units, and $\angle CAB = 60^\circ$. Find the length of the side AC .

Solution: Let $AC = x$. By the Law of Cosines,

$$81 = x^2 + 100 - 2 \cdot 10 \cdot x \cos 60^\circ, \quad x^2 - 10x + 19 = 0, \quad x = 5 \pm \sqrt{6}$$

Both roots are positive, and both triangles exist (the given angle is opposite the shorter given side). So $AC = 5 + \sqrt{6} \approx 7.45$ or $AC = 5 - \sqrt{6} \approx 2.55$.

2. The equations $x^2 + 3px + q = 0$ and $x^2 + px - q = 0$, where p and q are real parameters, have a common real root. Prove that the other two roots either add up to zero, or one of them is three times the other.

Solution: Let x_1 be the common root, and x_2, x_3 the other roots of the two equations. By Viète's formulas

$$x_1 + x_2 = -3p, \quad x_1 x_2 = q, \quad x_1 + x_3 = -p, \quad x_1 x_3 = -q$$

Adding the two products: $x_1(x_2 + x_3) = 0$. So either $x_2 + x_3 = 0$, or $x_1 = 0$; in the second case $x_2 = -3p$ and $x_3 = -p$, so $x_2 = 3x_3$.

3. Solve the following equation over the real numbers.

$$\cos 2x + \sqrt{\frac{1 - \sin 2x}{2}} = 0$$

Solution: The square root is nonnegative, so we need $\cos 2x \leq 0$. Then the equation is $\sqrt{\frac{1 - \sin 2x}{2}} = -\cos 2x$, and squaring (an equivalent step under this condition):

$$1 - \sin 2x = 2(1 - \sin^2 2x), \quad 2 \sin^2 2x - \sin 2x - 1 = 0, \quad \sin 2x = 1 \text{ or } \sin 2x = -\frac{1}{2}$$

If $\sin 2x = 1$, then $\cos 2x = 0$, and these are solutions: $x = \frac{\pi}{4} + k\pi$. If $\sin 2x = -\frac{1}{2}$ and $\cos 2x \leq 0$, then

$2x = \frac{7\pi}{6} + 2n\pi$: $x = \frac{7\pi}{12} + n\pi$. The solutions are

$$x = \frac{\pi}{4} + k\pi \quad \text{or} \quad x = \frac{7\pi}{12} + n\pi, \quad k, n \text{ are integers}$$

4. The base of the isosceles triangle ABC is $BC = 27$ units, and the radius of its inscribed circle is $\varrho = 9$ units. Find the radius of the circle that is tangent to the inscribed circle and to the legs.

Solution: Let O be the center of the inscribed circle, A_1 the midpoint of BC , and $m = AA_1$ the height. BO bisects the angle ABA_1 , so by the angle bisector theorem $\frac{BA}{OA} = \frac{BA_1}{OA_1} = \frac{13.5}{9} = \frac{3}{2}$. Let $BA = 3x$ and $OA = 2x$. In the right triangle ABA_1 :

$$(3x)^2 = 13.5^2 + (2x + 9)^2, \quad 5x^2 - 36x - 263.25 = 0, \quad x = 11.7$$

So $m = 2x + 9 = 32.4$. The tangent to the inscribed circle parallel to BC (at the top of the circle) cuts off a triangle similar to ABC , with height $m_1 = m - 2\varrho = 14.4$; the circle we are looking for is the inscribed circle of this small triangle. By the similarity,

$$\varrho_1 = \varrho \cdot \frac{m_1}{m} = 9 \cdot \frac{14.4}{32.4} = 4$$

5. Consider the circles $(x - 2)^2 + (y + 1)^2 = 16$ and $(x - 5)^2 + (y - 5)^2 = 1$.

- Find the coordinates of the intersection point of the external common tangents of the circles.
- Find the equations of the common tangents.

Solution: The centers are $K_1(2, -1)$ and $K_2(5, 5)$, and the radii are 4 and 1; $K_1K_2 = \sqrt{45} > 5$, so the circles are outside each other and have four common tangents.

- The external tangents meet at the external center of similarity H of the circles, on the line K_1K_2 beyond K_2 , with $\frac{HK_1}{HK_2} = \frac{4}{1}$. So $\vec{K_2H} = \frac{1}{3}\vec{K_1K_2} = (1, 2)$, and $H = (6, 7)$.

- The external tangents are the lines through H that are 1 unit away from K_2 . A line through H is $A(x - 6) + B(y - 7) = 0$, and the condition is

$$\frac{|-A - 2B|}{\sqrt{A^2 + B^2}} = 1, \quad 4AB + 3B^2 = 0$$

so $B = 0$ or $4A = -3B$. The external tangents are

$$x = 6 \quad \text{and} \quad 3x - 4y + 10 = 0$$

(Both are 4 units away from K_1 too.) The internal tangents go through the internal center of similarity $\left(\frac{22}{5}, \frac{19}{5}\right)$, and a similar computation gives

$$y = \frac{5\sqrt{5} - 9}{8}x + \frac{35 - 11\sqrt{5}}{4} \quad \text{and} \quad y = -\frac{5\sqrt{5} + 9}{8}x + \frac{35 + 11\sqrt{5}}{4}$$

(The solution sketch in KoMaL gives only the two external tangents.)

6. Which arithmetic sequences have the property that the quotient of the sum of the first $6n$ terms and the sum of the first $2n$ terms is the same for every positive integer n ?

Solution: Let the first term be a and the difference d , and suppose $\frac{S_{6n}}{S_{2n}} = K$ for every n :

$$\frac{6n}{2} (2a + (6n - 1)d) = K \cdot \frac{2n}{2} (2a + (2n - 1)d), \quad 2nd(K - 9) = (2a - d)(3 - K)$$

The right-hand side does not depend on n , so $d(K - 9) = 0$, and then $(2a - d)(3 - K) = 0$.

- If $d = 0$ (and $a \neq 0$): $K = 3$, the constant sequences $a, a, a, \dots$
- If $d \neq 0$: $K = 9$ and $d = 2a \neq 0$, the sequences $a, 3a, 5a, \dots$ (then $S_k = ak^2$, so the quotient is 9).

7. Solve the equation

$$\log_{\frac{1}{3}}(9^x + a) + \log_3(2 \cdot 3^x) = 0$$

over the real numbers, where a is a real parameter.

Solution: Since $\log_3(2 \cdot 3^x) = -\log_{\frac{1}{3}}(2 \cdot 3^x)$, the equation means $9^x + a = 2 \cdot 3^x$ (and then $9^x + a > 0$ automatically).

With $u = 3^x > 0$: $u^2 - 2u + a = 0$, so $u = 1 \pm \sqrt{1 - a}$.

- $a > 1$: no solution.
- $a = 1$: $u = 1$, the only solution is $x = 0$.
- $0 < a < 1$: both roots are positive: $x = \log_3(1 + \sqrt{1 - a})$ and $x = \log_3(1 - \sqrt{1 - a})$.
- $a \leq 0$: only $u = 1 + \sqrt{1 - a}$ is positive: $x = \log_3(1 + \sqrt{1 - a})$.

8. Solve the following equation over the integers.

$$\sin\left(\frac{\pi}{3}\left(x - \sqrt{x^2 - 3x - 12}\right)\right) = 0$$

Solution: The equation means $\frac{\pi}{3}\left(x - \sqrt{x^2 - 3x - 12}\right) = k\pi$, that is, $x - 3k = \sqrt{x^2 - 3x - 12}$ for some integer k . So we need $x - 3k \geq 0$, and squaring:

$$x^2 - 6kx + 9k^2 = x^2 - 3x - 12, \quad x = \frac{9k^2 + 12}{6k - 3} = \frac{3k^2 + 4}{2k - 1} = \frac{1}{4}\left(6k + 3 + \frac{19}{2k - 1}\right)$$

For x to be an integer, $2k - 1$ must divide 19: $2k - 1 = \pm 1$ or ± 19 .

$2k - 1 = 1$: $k = 1$, $x = 7$, and $x - 3k = 4 = \sqrt{49 - 21 - 12}$: a solution.

$2k - 1 = -1$: $k = 0$, $x = -4$, but $\sqrt{16 + 12 - 12} = 4 \neq -4$: not a solution.

$2k - 1 = 19$: $k = 10$, $x = 16$, but $x - 3k < 0$: not a solution.

$2k - 1 = -19$: $k = -9$, $x = -13$, and $x - 3k = 14 = \sqrt{169 + 39 - 12}$: a solution.

The integer solutions are $x = 7$ and $x = -13$.