

author: Imre Rábai

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In triangle ABC , $AC = 26$ units, and the medians AA_1 and CC_1 are 36 and 15 units long. Find the area of the triangle and the length of the median BB_1 .

Solution: Let S be the centroid; it divides each median in the ratio $2 : 1$, so $AS = 24$ and $CS = 10$. Since $24^2 + 10^2 = 26^2$, the triangle ASC has a right angle at S , and its area is $\frac{24 \cdot 10}{2} = 120$. The triangle ASC has one third of the area of ABC (its height from S is one third of the height from B), so the area of ABC is 360 square units.

SB_1 is the median to the hypotenuse of the right triangle ASC , so $SB_1 = \frac{AC}{2} = 13$, and $BB_1 = 3 \cdot SB_1 = 39$ units.

2. Let a, b, c be the sides of a triangle, and α, β, γ the opposite angles (as usual). Prove that the triangle is a right triangle if and only if

$$ab \cos \gamma + bc \cos \alpha + ca \cos \beta = d^2$$

where d is one of the sides a, b, c .

Solution: By the Law of Cosines, $ab \cos \gamma = \frac{a^2 + b^2 - c^2}{2}$, and similarly for the other two terms, so

$$ab \cos \gamma + bc \cos \alpha + ca \cos \beta = \frac{(a^2 + b^2 - c^2) + (b^2 + c^2 - a^2) + (c^2 + a^2 - b^2)}{2} = \frac{a^2 + b^2 + c^2}{2}$$

If this is c^2 , then $a^2 + b^2 = c^2$, and the triangle has a right angle at C (by the converse of the Pythagorean theorem); similarly if it is a^2 or b^2 . Conversely, if the triangle is a right triangle with hypotenuse c , then $a^2 + b^2 = c^2$, so $\frac{a^2 + b^2 + c^2}{2} = c^2$.

3. Solve the following inequalities over the real numbers.

$$\text{a) } 3^x - 3^{\frac{1}{2}-x} > \sqrt{3} - 1 \quad \text{b) } \frac{\sin^2 x + \frac{1}{2} \sin 2x}{\cos^2 x} > \sqrt{3} (\tan x + 1)$$

Solution: a) Let $y = 3^x > 0$; then $3^{\frac{1}{2}-x} = \frac{\sqrt{3}}{y}$, and multiplying by $y > 0$:

$$y^2 - (\sqrt{3} - 1)y - \sqrt{3} > 0, \quad (y + 1)(y - \sqrt{3}) > 0$$

Since $y + 1 > 0$, this means $3^x > \sqrt{3}$, that is, $x > \frac{1}{2}$.

b) We need $\cos x \neq 0$. The left-hand side is $\tan^2 x + \tan x$, so the inequality is

$$\tan^2 x + \tan x - \sqrt{3} \tan x - \sqrt{3} > 0, \quad (\tan x + 1)(\tan x - \sqrt{3}) > 0$$

that is, $\tan x < -1$ or $\tan x > \sqrt{3}$. The solution is

$$\frac{\pi}{3} + k\pi < x < \frac{\pi}{2} + k\pi \quad \text{or} \quad \frac{\pi}{2} + k\pi < x < \frac{3\pi}{4} + k\pi, \quad k \text{ is an integer}$$

4. The product of the second and sixth terms of a geometric sequence is 1, and the difference of the third and fifth terms is $\frac{3}{2}$. Find the first term and the ratio of the sequence.

Solution: Let the first term be a and the ratio $q \neq 0$. Then $aq \cdot aq^5 = (aq^3)^2 = 1$, so $aq^3 = \pm 1$, and $aq^2 - aq^4 = aq^3 \left(\frac{1}{q} - q \right) = \frac{3}{2}$.

If $aq^3 = 1$: $\frac{1}{q} - q = \frac{3}{2}$, $2q^2 + 3q - 2 = 0$, so $q = -2$ (then $a = -\frac{1}{8}$) or $q = \frac{1}{2}$ (then $a = 8$).

If $aq^3 = -1$: $q - \frac{1}{q} = \frac{3}{2}$, $2q^2 - 3q - 2 = 0$, so $q = 2$ (then $a = -\frac{1}{8}$) or $q = -\frac{1}{2}$ (then $a = 8$).

There are four such sequences: $(a, q) = \left(-\frac{1}{8}, -2\right), \left(8, \frac{1}{2}\right), \left(-\frac{1}{8}, 2\right), \left(8, -\frac{1}{2}\right)$.

5. Solve the following equation over the real numbers.

$$\log_2(x + 1) + \frac{1}{2} \log_2(x^2 - 8x + 16) = 1 + \log_2 3$$

Solution: We need $x > -1$ and $x \neq 4$. Since $x^2 - 8x + 16 = (x - 4)^2$, $\frac{1}{2} \log_2(x - 4)^2 = \log_2 |x - 4|$, and the equation is $\log_2((x + 1)|x - 4|) = \log_2 6$, that is, $(x + 1)|x - 4| = 6$.

If $x > 4$: $x^2 - 3x - 10 = 0$, so $x = 5$ (the root -2 is not allowed).

If $-1 < x < 4$: $-(x^2 - 3x - 4) = 6$, that is, $x^2 - 3x + 2 = 0$, so $x = 1$ or $x = 2$.

The solutions are $x = 1, x = 2, x = 5$.

6. Which points of the plane have the property that the tangent segments drawn from them to the circles $(x - 2)^2 + (y - 3)^2 = 36$ and $x^2 + (y - 1)^2 = 16$ are both 7 units long?

Solution: A tangent segment of length 7 from P to a circle with center K and radius r means $PK^2 = r^2 + 49$. So for $P(x, y)$:

$$(x - 2)^2 + (y - 3)^2 = 85, \quad x^2 + (y - 1)^2 = 65$$

Subtracting: $-4x + 4 - 4y + 8 = 20$, that is, $y = -x - 2$. Then $x^2 + (x + 3)^2 = 65$, $x^2 + 3x - 28 = 0$, so $x = 4$ or $x = -7$. The points are $P_1(4, -6)$ and $P_2(-7, 5)$.

7. Solve the following equation over the pairs of real numbers.

$$2 \sin^2 \frac{y + x^2}{4} = 3^x + 3^{-x}$$

Solution: The left-hand side is at most 2. The right-hand side is a positive number plus its reciprocal, so it is at least 2, with equality only if $3^x = 1$, that is, $x = 0$. So both sides are 2: $x = 0$ and $\sin^2 \frac{y}{4} = 1$, that is, $\frac{y}{4} = \frac{\pi}{2} + k\pi$. The solutions are

$$x = 0, \quad y = 2\pi + 4k\pi, \quad k \text{ is an integer}$$

8. Prove that if n is a positive integer, then

- a) 16 is a divisor of $5^n(4n - 1) + 1$,
b) 6000 is a divisor of $(25 \cdot 2^n - 2^n)(n^4 + 2n^3 - n^2 - 2n)$.

Solution: a) By the binomial theorem, $5^n = (1 + 4)^n = 1 + 4n + 16 \cdot \left(\binom{n}{2} + 4 \binom{n}{3} + \cdots \right)$, so $5^n = 1 + 4n + 16K$ for some integer K . Then

$$5^n(4n - 1) + 1 = (1 + 4n)(4n - 1) + 16K(4n - 1) + 1 = 16n^2 + 16K(4n - 1)$$

which is divisible by 16.

b) $n^4 + 2n^3 - n^2 - 2n = n^2(n^2 - 1) + 2n(n^2 - 1) = (n - 1)n(n + 1)(n + 2)$ is the product of four consecutive integers, so it is divisible by 3 and by 8 (among four consecutive integers there is a multiple of 3, and two consecutive even numbers, one of which is a multiple of 4), that is, by 24. And $25 \cdot 2^n - 2^n$ is divisible by $25 - 2 = 23$ (since $a^n - b^n$ is divisible by $a - b$). So the product is divisible by $23 \cdot 24 = 552$.