

author: László Számádó

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the real numbers.

$$\frac{3}{2x - 3y + 1} + \frac{4}{3x + 2y - 4} = \frac{13}{42}, \quad \frac{6}{3y - 2x - 1} - \frac{7}{4 - 3x - 2y} = -\frac{1}{12}$$

Solution: We need $2x - 3y + 1 \neq 0$ and $3x + 2y - 4 \neq 0$. Let $a = \frac{1}{2x - 3y + 1}$ and $b = \frac{1}{3x + 2y - 4}$. The denominators in the second equation are the negatives of these, so the system is

$$3a + 4b = \frac{13}{42}, \quad -6a + 7b = -\frac{1}{12}$$

Adding twice the first equation to the second one: $15b = \frac{45}{84}$, so $b = \frac{1}{28}$, and then $3a = \frac{13}{42} - \frac{1}{7} = \frac{1}{6}$, $a = \frac{1}{18}$. So $2x - 3y + 1 = 18$ and $3x + 2y - 4 = 28$, that is, $2x - 3y = 17$ and $3x + 2y = 32$, which gives

$$x = 10, \quad y = 1$$

2. We can cut the longer leg of a right triangle into two pieces so that the two pieces as legs, and the difference of the hypotenuse and the other leg as hypotenuse, again form a right triangle. Express the altitude to the hypotenuse of the new right triangle in terms of the sides of the original triangle.

Solution: Let the legs be $x > y$ and the hypotenuse z , and let the pieces be a and b (so $x = a + b$), with $c = z - y$. Then $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$, so

$$(a + b)^2 + y^2 = (y + c)^2, \quad a^2 + 2ab + b^2 = 2yc + c^2, \quad ab = yc$$

The area of the new triangle is $\frac{ab}{2} = \frac{c \cdot m}{2}$, where m is the altitude to the hypotenuse c . So $m = \frac{ab}{c} = y$: the altitude is equal to the shorter leg of the original triangle.

3. Find the radius of the circle circumscribed around a regular 12-gon with side 1.

Solution: Each side subtends a central angle of 30° , so the side is the base of an isosceles triangle with legs r and apex angle 30° : $\frac{1}{2} = r \sin 15^\circ$. Since $\sin 15^\circ = \sin (45^\circ - 30^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$,

$$r = \frac{1}{2 \sin 15^\circ} = \frac{2}{\sqrt{6} - \sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{2} \approx 1.932$$

4. Let n be a positive integer. For which real numbers x is the following equation true?

$$\frac{2}{\log_{3^1} x} + \frac{2}{\log_{3^2} x} + \cdots + \frac{2}{\log_{3^n} x} = (n^2 + n) \log_x 3$$

Solution: We need $x > 0$ and $x \neq 1$. Since $\frac{1}{\log_{3^k} x} = \log_x 3^k = k \log_x 3$, the left-hand side is

$$2(1 + 2 + \cdots + n) \log_x 3 = n(n+1) \log_x 3$$

which is the right-hand side. So the equation is true for every $x > 0$, $x \neq 1$.

5. Find the perimeter and the area of the polygon whose vertices are the common points of the circle with center $K(-2, 3)$ and radius $r = 5$, and the parabola with directrix $y = -\frac{41}{4}$ and focus $F\left(-2, -\frac{39}{4}\right)$.

Solution: The circle is $(x+2)^2 + (y-3)^2 = 25$. The vertex of the parabola is halfway between the focus and the directrix, at $(-2, -10)$, and its distance from the focus is $\frac{1}{4}$, so the parabola is $(x+2)^2 = y+10$. Substituting into the equation of the circle:

$$y+10+(y-3)^2=25, \quad y^2-5y-6=0, \quad y=-1 \text{ or } y=6$$

$y = -1$ gives $(x+2)^2 = 9$, and $y = 6$ gives $(x+2)^2 = 16$. The common points are $A(-5, -1)$, $B(1, -1)$, $C(2, 6)$, $D(-6, 6)$, which form an isosceles trapezoid with bases $AB = 6$, $CD = 8$, height 7, and legs $\sqrt{1+49} = 5\sqrt{2}$. Its perimeter is $14 + 10\sqrt{2} \approx 28.14$, and its area is $\frac{6+8}{2} \cdot 7 = 49$.

6. We add up the first ten terms of an arithmetic sequence, then the next five, and then the next four. The three numbers obtained are consecutive terms of another arithmetic sequence. Find the quotient of the first term and the difference of the original sequence.

Solution: Let the first term be a_1 and the difference d . The three sums are

$$b_1 = S_{10} = 10a_1 + 45d, \quad b_2 = S_{15} - S_{10} = 5a_1 + 60d, \quad b_3 = S_{19} - S_{15} = 4a_1 + 66d$$

They form an arithmetic sequence if $2b_2 = b_1 + b_3$: $10a_1 + 120d = 14a_1 + 111d$, that is, $9d = 4a_1$. (If $d = 0$, then also $a_1 = 0$.) So $\frac{a_1}{d} = \frac{9}{4}$.

7. Depending on the parameter p , find the number of solutions of the following equation.

$$x^2 - 2x + 7 = px + \left(5 - \frac{p^3}{4}\right)$$

Solution: The equation is $x^2 - (p+2)x + \frac{p^3}{4} + 2 = 0$, with discriminant

$$D = (p+2)^2 - p^3 - 8 = -p^3 + p^2 + 4p - 4 = (1-p)(p^2 - 4) = (1-p)(p-2)(p+2)$$

- $p < -2$ or $1 < p < 2$: $D > 0$, two solutions.
- $p = -2$, $p = 1$, or $p = 2$: $D = 0$, one solution.
- $-2 < p < 1$ or $p > 2$: $D < 0$, no real solution.

8. The numbers a and b are positive reals. Show that for every natural number n ,

$$(a+b+2)^{4n} \geq 256n^3ab$$

Solution: For $n = 0$ the inequality says $1 \geq 0$. Let $n \geq 1$. By Bernoulli's inequality ($(1+q)^n \geq 1+nq$ for $q > -1$, which is easy to prove by induction),

$$(a+b+2)^n = (1+(a+b+1))^n \geq 1+n(a+b+1) = 1+na+nb+n$$

By the inequality between the arithmetic and geometric means of the four positive numbers $1, na, nb, n$:

$$1+na+nb+n \geq 4\sqrt[4]{n^3ab}$$

Raising to the fourth power: $(a+b+2)^{4n} \geq 256n^3ab$.