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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the function  $f(x) = (x-1)(x-2)(x-3)(x-4) + 10$ , defined on the real numbers.

- a) Prove that  $f(x) > 0$  for every real  $x$ .

- b) Where does  $f(x)$  take its smallest value, and what is the smallest value?

Solution:  $(x-1)(x-4) = x^2 - 5x + 4$  and  $(x-2)(x-3) = x^2 - 5x + 6$ . With  $u = x^2 - 5x$ :

$$f(x) = (u+4)(u+6) + 10 = u^2 + 10u + 34 = (u+5)^2 + 9 = (x^2 - 5x + 5)^2 + 9$$

- a) So  $f(x) \geq 9 > 0$ .

- b) The smallest value is 9, where  $x^2 - 5x + 5 = 0$ , that is, at  $x = \frac{5 + \sqrt{5}}{2}$  and  $x = \frac{5 - \sqrt{5}}{2}$ .

2. One parallel side of the trapezoid  $ABCD$  is  $AB = 4.4$  units, the legs are  $BC = DA = 1.7$  units, and the diagonals are  $AC = BD = 3.9$  units. Find the area of the trapezoid.

Solution: The triangle  $ABC$  has sides 4.4, 1.7, 3.9; by Heron's formula with  $s = 5$ , its area is  $\sqrt{5 \cdot 0.6 \cdot 3.3 \cdot 1.1} = \sqrt{10.89} = 3.3$ . So the distance  $CC_1$  of  $C$  from  $AB$  (the height of the trapezoid) is  $\frac{2 \cdot 3.3}{4.4} = 1.5$ . Then  $AC_1 = \sqrt{3.9^2 - 1.5^2} = 3.6$  and  $C_1B = 0.8$ , so the other parallel side is  $DC = 4.4 - 2 \cdot 0.8 = 2.8$ . The area is

$$T = \frac{4.4 + 2.8}{2} \cdot 1.5 = 5.4 \text{ square units}$$

(In an isosceles trapezoid  $AC_1$  is equal to the midline, so  $T = AC_1 \cdot CC_1$ .)

3. The equations  $x^2 + px - q = 0$  and  $x^2 + qx - p = 0$  have one common root, and their other roots are different.

- a) Prove that the sum of the roots that are not common is 1.  
b) Using  $p$ , write down a quadratic equation whose roots are the two different roots in question.

Solution: Subtracting the second equation from the first one, the common root  $x_0$  satisfies  $(p - q)x_0 + (p - q) = 0$ . Here  $p \neq q$  (otherwise the two equations would be the same, and all their roots would be common), so  $x_0 = -1$ . Substituting into the first equation:  $1 - p - q = 0$ , so  $p + q = 1$ .

- a) By Viète's formulas, the other root of the first equation is  $x_1 = \frac{-q}{x_0} = q$ , and the other root of the second one is  $x_2 = \frac{-p}{x_0} = p$ . So  $x_1 + x_2 = p + q = 1$ .  
b) The roots are  $p$  and  $q = 1 - p$ , so a suitable equation is

$$x^2 - x + p(1 - p) = 0$$

4. Solve the following equation over the real numbers.

$$\frac{\sqrt{1 + \sin 2x}}{\sin x + \cos x} = -1$$

Solution:  $1 + \sin 2x = (\sin x + \cos x)^2$ , so the numerator is  $|\sin x + \cos x|$ , and the equation holds exactly when  $\sin x + \cos x < 0$ . Since  $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ , this means  $\pi + 2k\pi < x + \frac{\pi}{4} < 2\pi + 2k\pi$ :

$$\frac{3\pi}{4} + 2k\pi < x < \frac{7\pi}{4} + 2k\pi, \quad k \text{ is an integer}$$

5. Solve the following equation over the pairs of positive integers.

$$x^2 - 3xy + 2y^2 + 6 = 0$$

Solution: The equation can be written as  $(y - x)(x - 2y) = 6$ . Both factors are integers, so they are 1 and 6, 6 and 1, 2 and 3, 3 and 2, or the negatives of these. Solving each case (from  $y - x = u$  and  $x - 2y = v$  we get  $y = -u - v$ ,  $x = -2u - v$ ), and keeping only positive integers:

$$\begin{aligned} y - x = -1, x - 2y = -6 & : (x, y) = (8, 7) \\ y - x = -6, x - 2y = -1 & : (x, y) = (13, 7) \\ y - x = -2, x - 2y = -3 & : (x, y) = (7, 5) \\ y - x = -3, x - 2y = -2 & : (x, y) = (8, 5) \end{aligned}$$

(the other four cases give  $y < 0$ ). The solutions are  $(7, 5)$ ,  $(8, 5)$ ,  $(8, 7)$ ,  $(13, 7)$ .

6. In triangle  $ABC$  (with the usual notation)  $\gamma = 36^\circ$  and  $a^2 - b^2 = 2cr$ , where  $r$  is the radius of the circumscribed circle. Find the other two angles of the triangle.

Solution: By the Law of Sines  $a = 2r \sin \alpha$ ,  $b = 2r \sin \beta$ ,  $c = 2r \sin \gamma$ , so the condition is  $\sin^2 \alpha - \sin^2 \beta = \sin \gamma$ . The left-hand side is  $(\sin \alpha + \sin \beta)(\sin \alpha - \sin \beta) = \sin(\alpha + \beta) \sin(\alpha - \beta)$  (by the sum-to-product formulas), and  $\sin \gamma = \sin(\alpha + \beta) > 0$ . So  $\sin(\alpha - \beta) = 1$ , that is,  $\alpha - \beta = 90^\circ$ . With  $\alpha + \beta = 144^\circ$ :  $\alpha = 117^\circ$  and  $\beta = 27^\circ$ .

7. Reflect the circle  $x^2 + y^2 - 12x + 2y + 33 = 0$  in the line  $x - 2y = -2$ . Find the equations of the common tangents of the two circles.

Solution: The circle is  $(x - 6)^2 + (y + 1)^2 = 4$ , with center  $C(6, -1)$  and radius 2. The foot of the perpendicular from  $C$  to the line  $x - 2y + 2 = 0$  is  $T(4, 3)$ , and the reflected center is  $C'(2, 7)$ ;  $CC' = 4\sqrt{5} > 4$ , so the circles are outside each other and have four common tangents.

The external tangents are parallel to  $CC'$ , that is, perpendicular to the mirror line:  $2x + y + k = 0$ , at distance 2 from  $C$ :  $\frac{|11 + k|}{\sqrt{5}} = 2$ . So

$$2x + y - 11 + 2\sqrt{5} = 0 \quad \text{and} \quad 2x + y - 11 - 2\sqrt{5} = 0$$

The internal tangents go through the midpoint  $T(4, 3)$  of  $CC'$ :  $A(x - 4) + B(y - 3) = 0$  with  $\frac{|2A - 4B|}{\sqrt{A^2 + B^2}} = 2$ , that is,  $B(3B - 4A) = 0$ . So

$$x = 4 \quad \text{and} \quad 3x + 4y - 24 = 0$$

8. Solve the following equation over the real numbers, where  $a$  is a real parameter.

$$\frac{x - a}{x - 2} + \frac{10}{x + 2} = \frac{44}{4 - x^2}$$

Solution: We need  $x \neq \pm 2$ . Multiplying by  $x^2 - 4$ :

$$(x - a)(x + 2) + 10(x - 2) = -44, \quad x^2 + (12 - a)x + 2(12 - a) = 0$$

with discriminant  $D = (12 - a)^2 - 8(12 - a) = (a - 4)(a - 12)$ .  $x = -2$  is never a root (it gives  $4 = 0$ ), and  $x = 2$  is a root only for  $a = 13$  (then the other root is  $-1$ ).

- $4 < a < 12$ : no solution.
- $a = 4$ :  $x = -4$ .  $a = 12$ :  $x = 0$ .  $a = 13$ :  $x = -1$ .
- $a < 4$ , or  $a > 12$  and  $a \neq 13$ : two solutions,

$$x = \frac{a - 12 \pm \sqrt{(a - 4)(a - 12)}}{2}$$