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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The parallel sides of trapezoid $ABCD$ are $AB = 39$ and $DC = 26$ units long, and the legs are $BC = 12$ and $DA = 5$ units long. Find the area and the angles of the trapezoid.

Solution: The line through C parallel to DA meets AB at E ; then $AECD$ is a parallelogram, so $AE = 26$, $EC = 5$, and $EB = 13$. Since $5^2 + 12^2 = 13^2$, the triangle EBC has a right angle at C . The height m of the trapezoid is the altitude of this triangle to EB : $13m = 5 \cdot 12$, so $m = \frac{60}{13}$, and the area is

$$T = \frac{39 + 26}{2} \cdot \frac{60}{13} = 150 \text{ square units}$$

In the right triangle EBC , $\sin \angle CBE = \frac{5}{13}$ and $\sin \angle CEB = \frac{12}{13}$. So the angle at B is about 22.62° , the angle at A (equal to $\angle CEB$) is about 67.38° , and the angles at D and C are about 112.62° and 157.38° .

2. In triangle ABC , B_1 is the midpoint of the side AC , $AB = \sqrt{10}$, $BC = \sqrt{30}$ units, and $AC = 4BB_1$. Find the length of the side AC and the area of the triangle.

Solution: Let $BB_1 = s$ and $\varphi = \angle BB_1A$; then $AB_1 = B_1C = 2s$ and $\angle BB_1C = 180^\circ - \varphi$. By the Law of Cosines in the triangles AB_1B and CB_1B :

$$10 = s^2 + 4s^2 - 4s^2 \cos \varphi, \quad 30 = s^2 + 4s^2 + 4s^2 \cos \varphi$$

Adding: $40 = 10s^2$, so $s = 2$, and $AC = 4s = 8$ units. Then $\cos \varphi = \frac{5}{8}$, $\sin \varphi = \frac{\sqrt{39}}{8}$, and the area is

$$T = 2 \cdot \frac{1}{2} \cdot s \cdot 2s \cdot \sin \varphi = 8 \cdot \frac{\sqrt{39}}{8} = \sqrt{39} \approx 6.24 \text{ square units}$$

3. For four years, at the beginning of every year we deposit 20 000 Ft in a bank, at 25% annual compound interest.
- How much money will we have in the bank at the end of the fourth year?
 - How much money B would we have had to deposit at the beginning of the first year so that it grows to the same amount by the end of the fourth year as in part a)?

Solution: a) The deposits earn interest for 4, 3, 2, and 1 years:

$$20\,000 (1.25^4 + 1.25^3 + 1.25^2 + 1.25) = 20\,000 \cdot 1.25 \cdot \frac{1.25^4 - 1}{1.25 - 1} = 100\,000 (1.25^4 - 1) \approx 144\,141 \text{ Ft}$$

b) $B \cdot 1.25^4 = 144\,141$, so $B \approx 59\,040 \text{ Ft}$.

4. Solve the following equations over the real numbers.

a) $4^{\frac{4}{y}+2} = 16 \cdot 4^{\frac{3y}{2y+4}}$ b) $2 \sin^3 x = \cos 2x + \sin x$

Solution: a) We need $y \neq 0$, $y \neq -2$. The right-hand side is $4^{2+\frac{3y}{2y+4}}$, and the exponential function is one-to-one, so

$$\frac{4}{y} = \frac{3y}{2y+4}, \quad 3y^2 - 8y - 16 = 0, \quad y = 4 \text{ or } y = -\frac{4}{3}$$

b) The equation is $\cos 2x + \sin x (1 - 2 \sin^2 x) = 0$, and $1 - 2 \sin^2 x = \cos 2x$, so $\cos 2x (1 + \sin x) = 0$. The solutions are

$$x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi, \quad k, n \text{ are integers}$$

5. Find the equation of the line that goes through the point $P(2, 5)$ and is twice as far from the point $A(5, 3)$ as from the point $B(-1, 0)$.

Solution: A line through P is $A_0x + B_0y - 2A_0 - 5B_0 = 0$ (with $A_0^2 + B_0^2 > 0$). The condition is

$$|3A_0 - 2B_0| = 2|-3A_0 - 5B_0|$$

so $3A_0 - 2B_0 = \pm 2(3A_0 + 5B_0)$, which gives $A_0 : B_0 = 4 : (-1)$ or $8 : (-9)$. There are two such lines:

$$4x - y = 3 \quad \text{and} \quad 8x - 9y = -29$$

(They go through the points $C(1, 1)$ and $D(-7, -3)$, for which $\overrightarrow{AC} = 2\overrightarrow{CB}$ and $\overrightarrow{AD} = 2\overrightarrow{BD}$.)

6. Solve the following system of equations over the pairs of real numbers.

$$\log_3 x + 1 = \log_3 y, \quad x^{\log_3 y} + 2 \cdot y^{\log_3 x} = 27$$

Solution: We need $x > 0$, $y > 0$, and the first equation gives $y = 3x$. Then $x^{\log_3 y} = x^{1+\log_3 x}$ and $y^{\log_3 x} = 3^{\log_3 x} \cdot x^{\log_3 x} = x \cdot x^{\log_3 x}$, so the second equation is $3x \cdot x^{\log_3 x} = 27$, that is, $x^{1+\log_3 x} = 9$. Taking $\log_3$:

$$\log_3 x + (\log_3 x)^2 = 2, \quad \log_3 x = 1 \text{ or } \log_3 x = -2$$

The solutions are $(x, y) = (3, 9)$ and $\left(\frac{1}{9}, \frac{1}{3}\right)$.

7. Find the values of the real parameter k for which one root of the equation

$$x^2 - 2 \cdot \frac{k+3}{k-2}x + \frac{4k}{k-2} = 0$$

is less than 2, and the other one is greater than 3.

Solution: The left-hand side $f(x)$ is a quadratic function with positive leading coefficient, so this happens exactly when $f(2) < 0$ and $f(3) < 0$:

$$f(2) = \frac{4k-20}{k-2} < 0, \quad f(3) = \frac{7k-36}{k-2} < 0$$

The first gives $2 < k < 5$, the second $2 < k < \frac{36}{7}$. So $2 < k < 5$.

8. The sides of triangle ABC are $AB = 13$, $AC = 15$, $BC = 14$ units. A point P is chosen on the altitude from A . The line e through P parallel to BC meets the side AB at E and the side AC at F . On the line e , outside the triangle, we take the points B' and C' on the rays PE and PF so that $EB' = EA$ and $FC' = FA$. How far is P from A , if the area of the trapezoid $BCC'B'$ is as large as possible?

Solution: By Heron's formula the area of ABC is 84, so the altitude from A is $m = 12$. Let $AP = x$ ($0 < x < 12$). The triangle AEF is similar to ABC with ratio $\frac{x}{12}$, so its perimeter is $\frac{42x}{12} = \frac{7x}{2}$, and this is the length of $B'C' = B'E + EF + FC' = AE + EF + FA$. The trapezoid has bases 14 and $\frac{7x}{2}$ and height $12 - x$:

$$t(x) = \frac{\left(14 + \frac{7x}{2}\right)(12 - x)}{2} = \frac{7}{4}(4 + x)(12 - x) = \frac{7}{4}(64 - (x - 4)^2)$$

This is largest for $x = 4$, with area 112. So $AP = 4$ units: P divides the altitude into thirds, closer to A .