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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

There are 180 minutes to solve and present all problems.

1. For which x are the following functions defined, and what are their ranges?

$$f_1(x) = x^2 - 9x - 22, \quad f_2(x) = |f_1(x)|, \quad f_3(x) = \sqrt{f_1(x)}, \quad f_4(x) = \log f_1(x)$$

Solution: $f_1(x) = \left(x - \frac{9}{2}\right)^2 - \frac{169}{4} = (x - 11)(x + 2).$

f_1 : defined for every real x ; range: $f_1(x) \geq -\frac{169}{4}.$

f_2 : defined for every real x ; range: $f_2(x) \geq 0.$

f_3 : defined for $x \leq -2$ or $x \geq 11$; range: $f_3(x) \geq 0.$

f_4 : defined for $x < -2$ or $x > 11$; range: all real numbers.

2. The roots of the equation $x^2 - 10x + 21 = 0$ are the radii of the bases of two equilateral cylinders (cylinders whose height is equal to the diameter of the base). Without computing the roots, find the sum of the surface areas and the sum of the volumes of the cylinders.

Solution: Let the roots be r_1 and r_2 ; by Viète's formulas $r_1 + r_2 = 10$ and $r_1 r_2 = 21$ (both are positive). A cylinder with radius r and height $2r$ has surface area $2r^2\pi + 2r\pi \cdot 2r = 6r^2\pi$ and volume $r^2\pi \cdot 2r = 2r^3\pi$. So

$$A_1 + A_2 = 6\pi (r_1^2 + r_2^2) = 6\pi ((r_1 + r_2)^2 - 2r_1 r_2) = 6\pi (100 - 42) = 348\pi$$

$$V_1 + V_2 = 2\pi (r_1^3 + r_2^3) = 2\pi ((r_1 + r_2)^3 - 3r_1 r_2 (r_1 + r_2)) = 2\pi (1000 - 630) = 740\pi$$

3. Find the real numbers x for which

$$(\sqrt{x+1} - 2) \cdot (\sqrt{x+3} - 3) \cdot (\sqrt{x+5} - 4) \geq 0$$

Solution: We need $x \geq -1$. Multiplying by the positive number $(\sqrt{x+1} + 2)(\sqrt{x+3} + 3)(\sqrt{x+5} + 4)$ and using $(u - v)(u + v) = u^2 - v^2$ three times:

$$(x + 1 - 4)(x + 3 - 9)(x + 5 - 16) \geq 0, \quad (x - 3)(x - 6)(x - 11) \geq 0$$

A sign chart gives $3 \leq x \leq 6$ or $x \geq 11$, and these are in the domain.

4. The lines a, b, c with equations $2x + y + 9 = 0$, $-x + 2y + 8 = 0$, $3x + 4y + 6 = 0$ form a triangle. Find the equation of its circumscribed circle, and the radius of its inscribed circle.

Solution: The normal vectors of a and b are $(2, 1)$ and $(-1, 2)$, which are perpendicular, so the triangle has a right angle at the intersection C of a and b . The vertices are $C = a \cap b = (-2, -5)$, $A = b \cap c = (2, -3)$, $B = a \cap c = (-6, 3)$. The hypotenuse AB is a diameter of the circumscribed circle (Thales' theorem): its midpoint is $(-2, 0)$ and $AB = 10$, so the circle is

$$(x + 2)^2 + y^2 = 25$$

The legs are $CA = 2\sqrt{5}$ and $CB = 4\sqrt{5}$. The area is $\frac{2\sqrt{5} \cdot 4\sqrt{5}}{2} = 20$, and the semiperimeter is $5 + 3\sqrt{5}$, so the radius of the inscribed circle is

$$\rho = \frac{20}{5 + 3\sqrt{5}} = 3\sqrt{5} - 5 \approx 1.71$$

5. In a plane, of the points S, Z, L , the points S and L are fixed and Z is moving, with $SL = 2$, $SZ = s$, $LZ = l$, and $2s^2 + 4 = l^2$. What is the set of the points Z ?

Solution: Use coordinates with $S(-1, 0)$, $L(1, 0)$, and $Z(x, y)$. Then $s^2 = (x + 1)^2 + y^2$ and $l^2 = (x - 1)^2 + y^2$, and the condition is

$$2((x + 1)^2 + y^2) + 4 = (x - 1)^2 + y^2, \quad x^2 + y^2 + 6x + 5 = 0, \quad (x + 3)^2 + y^2 = 4$$

So the points Z form the circle with radius 2 whose center is on the line LS , beyond S , at distance 2 from S (that is, at the reflection of L in S). (Every point of this circle satisfies the condition, since all steps can be reversed.)

6. Solve the following equation.

$$36 \sin^4 x - 12 \sin^3 x - 17 \sin^2 x + 3 \sin x + 2 = 0$$

Solution: Writing $-17 \sin^2 x = \sin^2 x - 18 \sin^2 x$, and with $u = 6 \sin^2 x - \sin x$, the equation is

$$(6 \sin^2 x - \sin x)^2 - 3(6 \sin^2 x - \sin x) + 2 = 0, \quad (u - 1)(u - 2) = 0$$

$6 \sin^2 x - \sin x = 1$ gives $\sin x = \frac{1}{2}$ or $\sin x = -\frac{1}{3}$, and $6 \sin^2 x - \sin x = 2$ gives $\sin x = \frac{2}{3}$ or $\sin x = -\frac{1}{2}$. The solutions are (with k any integer)

$$\begin{aligned} x &= 30^\circ + k \cdot 360^\circ, \quad 150^\circ + k \cdot 360^\circ, \quad -30^\circ + k \cdot 360^\circ, \quad 210^\circ + k \cdot 360^\circ, \\ x &\approx 41.81^\circ + k \cdot 360^\circ, \quad 138.19^\circ + k \cdot 360^\circ, \quad -19.47^\circ + k \cdot 360^\circ, \quad 199.47^\circ + k \cdot 360^\circ \end{aligned}$$

7. Is it true that the system of equations $xy = c$, $x - y = c$ has a solution over the rational numbers if and only if c is the reciprocal of the product of two consecutive integers (neither of them 0)?

Solution: No, only one direction is true.

If $c = \frac{1}{n(n+1)}$, then substituting $x = y + c$ into $xy = c$ gives $y^2 + cy - c = 0$, whose discriminant is

$$c^2 + 4c = \frac{1 + 4n(n+1)}{n^2(n+1)^2} = \left(\frac{2n+1}{n(n+1)} \right)^2$$

a perfect square of a rational number, so y (and x) are rational.

But the converse is false: for example, $x = 3$, $y = \frac{3}{4}$ is a rational solution for $c = \frac{9}{4}$ (since $3 \cdot \frac{3}{4} = \frac{9}{4} = 3 - \frac{3}{4}$), and $\frac{9}{4}$ is not the reciprocal of a product of two consecutive integers.

8. The volume and the surface area of a right circular cone are in the ratio 7 : 4 (as numbers). Find the radius of the base, the height, and the slant height of the cone, if they are integers.

Solution: Let the radius be r , the height m , and the slant height a . Then

$$\frac{\frac{1}{3}r^2\pi m}{r\pi(a+r)} = \frac{7}{4}, \quad 4rm = 21(a+r), \quad 4rm - 21r = 21\sqrt{r^2 + m^2}$$

Squaring and simplifying: $16r^2m - 168r^2 - 441m = 0$, that is, $(4r)^2(2m - 21) = 882m$, so $m \geq 11$ and

$$(4r)^2 = \frac{882m}{2m - 21} = 441 + \frac{9261}{2m - 21}$$

So $2m - 21$ is a positive divisor of $9261 = 3^3 \cdot 7^3$, and $441 + \frac{9261}{2m - 21}$ must be the square of a multiple of 4. Checking the 16 divisors, only $\frac{9261}{2m - 21} = 343$ works: $441 + 343 = 784 = 28^2$. Then $r = 7$, $2m - 21 = 27$, $m = 24$, and $a = \sqrt{49 + 576} = 25$. The cone has radius 7, height 24, and slant height 25.