

Problem Set II - 1996 (December 1996)

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In memory of Elemér Simon, teacher (Elemér Simon was the author's mathematics teacher at the Klauzál Gábor High School in Szeged.)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. a) The first term of a geometric sequence is 2, the sum of the first n terms is 16, and the sum of the reciprocals of the first n terms is 4. Write down the first n terms of the sequence.

- b) The first term of a geometric sequence is $\frac{1}{2}$, the sum of the first n terms is 20, and the sum of the reciprocals of the first n terms is $\frac{80}{27}$. Write down the first n terms of the sequence.

2. The sides of a triangle are $a = 6.8$, $b = 22.1$, $c = 25.5$ units. Find

- a) the area of the triangle,
b) the radius of its circumscribed circle,
c) the radius of its inscribed circle.

3. Solve the following system of equations over the triples of real numbers.

$$x + y = 2z^2, \quad y + z = 2x^2, \quad z + x = 2y^2$$

4. Prove that if a , b , c are the sides of a triangle and α , β , γ are the opposite angles, then

$$a^2 + b^2 - (\sqrt{6} + \sqrt{2}) ab \cos(\gamma + 75^\circ) = b^2 + c^2 - (\sqrt{6} + \sqrt{2}) bc \cos(\alpha + 75^\circ)$$

Write an expression in terms of c , α , and β that is equal to these two expressions.

5. Consider the function

$$x \mapsto f(x) = (\log_2 x)^4 + 12 (\log_2 x)^2 \cdot \log_2 \frac{8}{x}$$

defined for the real numbers $1 < x < 64$. Find the largest value of the function. Where does the function take this largest value?

6. At the beginning of the year we took out a loan of 100 000 Ft from a bank for six months. We pay it back in six installments, at the end of each month, starting at the end of the first month. The bank charges 2.5% monthly compound interest in the first three months, and 2% monthly compound interest in the next three months. The monthly installments are equal in the first three months, and also equal in the second three months, but twice as large as in the first three months. Find the installments.

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7. The equations of the lines of two sides of a rhombus are $x - y - 7 = 0$ and $x + 7y - 31 = 0$. Find the equation of the circle inscribed in the rhombus, if its radius is $\varrho = 2\sqrt{2}$ units.
8. Solve the equation

$$\sqrt{x^2 + x + \frac{p^2}{(x-1)^2}} = x - \frac{p}{x-1}$$

over the real numbers, where p is a real parameter. For which values of p does the equation have two different solutions?