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In memory of Elemér Simon, teacher (Elemér Simon was the author's mathematics teacher at the Klauzál Gábor High School in Szeged.)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. a) The first term of a geometric sequence is 2, the sum of the first n terms is 16, and the sum of the reciprocals of the first n terms is 4. Write down the first n terms of the sequence.
- b) The first term of a geometric sequence is $\frac{1}{2}$, the sum of the first n terms is 20, and the sum of the reciprocals of the first n terms is $\frac{80}{27}$. Write down the first n terms of the sequence.

Solution: For a geometric sequence with terms $a_1, \dots, a_n$, the sum S and the sum of the reciprocals R satisfy $\frac{S}{R} = a_1 a_n$, since

$$R = \frac{1}{a_1} + \dots + \frac{1}{a_n} = \frac{a_n + a_{n-1} + \dots + a_1}{a_1 a_n}$$

(because $a_k a_{n+1-k} = a_1 a_n$).

a) $2a_n = \frac{16}{4} = 4$, so $a_n = 2 = a_1$, that is, $q^{n-1} = 1$, so $q = 1$ or $q = -1$. With $q = -1$ the sum would be 0 or 2, so $q = 1$, and $2n = 16$: the sequence is eight 2's: 2, 2, 2, 2, 2, 2, 2, 2.

b) $\frac{a_n}{2} = \frac{20 \cdot 27}{80} = \frac{27}{4}$, so $a_n = \frac{27}{2}$ and $q^{n-1} = 27$ (so $q \neq 1$). The sum is $\frac{1}{2} \cdot \frac{q^n - 1}{q - 1} = \frac{1}{2} \cdot \frac{27q - 1}{q - 1} = 20$, so $27q - 1 = 40q - 40$, $q = 3$, and $n = 4$. The terms are $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}$.

2. The sides of a triangle are $a = 6.8$, $b = 22.1$, $c = 25.5$ units. Find

- a) the area of the triangle,
- b) the radius of its circumscribed circle,
- c) the radius of its inscribed circle.

Solution: a) $s = 27.2$, so by Heron's formula $T^2 = 27.2 \cdot 20.4 \cdot 5.1 \cdot 1.7 = 69.36^2$, and $T = 69.36$ square units.

b) $T = \frac{abc}{4r}$, so $r = \frac{abc}{4T} = \frac{6.8 \cdot 22.1 \cdot 25.5}{4 \cdot 69.36} = \frac{221}{16} = 13.8125$ units.

c) $T = \varrho s$, so $\varrho = \frac{69.36}{27.2} = 2.55$ units.

3. Solve the following system of equations over the triples of real numbers.

$$x + y = 2z^2, \quad y + z = 2x^2, \quad z + x = 2y^2$$

Solution: Subtracting the second equation from the first one: $x - z = 2(z^2 - x^2)$, that is, $(x - z)(1 + 2x + 2z) = 0$. But $2x + 2z + 1 = 0$ is impossible, because then $z + x = -\frac{1}{2}$, while by the third equation $z + x = 2y^2 \geq 0$. So $x = z$. In the same way, from the second and the third equations, $(y - x)(1 + 2x + 2y) = 0$, and $x + y = -\frac{1}{2}$ is impossible by the first equation, so $x = y$. Then $2x = 2x^2$, so $x = 0$ or $x = 1$. The solutions are $(0, 0, 0)$ and $(1, 1, 1)$.

4. Prove that if a, b, c are the sides of a triangle and α, β, γ are the opposite angles, then

$$a^2 + b^2 - (\sqrt{6} + \sqrt{2}) ab \cos(\gamma + 75^\circ) = b^2 + c^2 - (\sqrt{6} + \sqrt{2}) bc \cos(\alpha + 75^\circ)$$

Write an expression in terms of c, α , and β that is equal to these two expressions.

Solution: $\cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ and $\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$, so

$$(\sqrt{6} + \sqrt{2}) \cos(\gamma + 75^\circ) = \frac{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}{4} \cos \gamma - \frac{(\sqrt{6} + \sqrt{2})^2}{4} \sin \gamma = \cos \gamma - (2 + \sqrt{3}) \sin \gamma$$

So, with T the area of the triangle, $ab \cos \gamma = \frac{a^2 + b^2 - c^2}{2}$ and $ab \sin \gamma = 2T$:

$$a^2 + b^2 - (\sqrt{6} + \sqrt{2}) ab \cos(\gamma + 75^\circ) = a^2 + b^2 - \frac{a^2 + b^2 - c^2}{2} + (4 + 2\sqrt{3}) T = \frac{a^2 + b^2 + c^2}{2} + (4 + 2\sqrt{3}) T$$

This is symmetric in the sides, so the second expression (the same with b, c, α in place of a, b, γ) has the same value.

To write it in terms of c, α, β , use the Law of Sines, $a = \frac{c \sin \alpha}{\sin(\alpha + \beta)}$, $b = \frac{c \sin \beta}{\sin(\alpha + \beta)}$, and $T = \frac{c^2 \sin \alpha \sin \beta}{2 \sin(\alpha + \beta)}$:

$$\frac{c^2}{2} \cdot \frac{\sin^2 \alpha + \sin^2 \beta + \sin^2(\alpha + \beta)}{\sin^2(\alpha + \beta)} + (2 + \sqrt{3}) \frac{c^2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

5. Consider the function

$$x \mapsto f(x) = (\log_2 x)^4 + 12 (\log_2 x)^2 \cdot \log_2 \frac{8}{x}$$

defined for the real numbers $1 < x < 64$. Find the largest value of the function. Where does the function take this largest value?

Solution: Let $t = \log_2 x$; then $0 < t < 6$, and $\log_2 \frac{8}{x} = 3 - t$. So

$$f(x) = t^4 - 12t^3 + 36t^2 = (t^2 - 6t)^2 = ((t - 3)^2 - 9)^2$$

For $0 < t < 6$, $-9 \leq (t - 3)^2 - 9 < 0$, so $f(x) \leq 81$, with equality for $t = 3$. The largest value is 81, at $x = 8$.

6. At the beginning of the year we took out a loan of 100 000 Ft from a bank for six months. We pay it back in six installments, at the end of each month, starting at the end of the first month. The bank charges 2.5% monthly compound interest in the first three months, and 2% monthly compound interest in the next three months. The monthly installments are equal in the first three months, and also equal in the second three months, but twice as large as in the first three months. Find the installments.

Solution: Let the first installments be x , and $q_1 = 1.025$, $q_2 = 1.02$. After three months the debt is $K = 100\,000q_1^3 - x(q_1^2 + q_1 + 1)$, and after six months it is $Kq_2^3 - 2x(q_2^2 + q_2 + 1) = 0$. So

$$x = \frac{100\,000q_1^3q_2^3}{q_2^3(q_1^2 + q_1 + 1) + 2(q_2^2 + q_2 + 1)} \approx 12\,177.35$$

The installments are about 12 177 Ft in the first three months and about 24 355 Ft in the second three months.

7. The equations of the lines of two sides of a rhombus are $x - y - 7 = 0$ and $x + 7y - 31 = 0$. Find the equation of the circle inscribed in the rhombus, if its radius is $\rho = 2\sqrt{2}$ units.

Solution: The center of the inscribed circle is $2\sqrt{2}$ away from both lines:

$$\frac{|x - y - 7|}{\sqrt{2}} = 2\sqrt{2}, \quad \frac{|x + 7y - 31|}{\sqrt{50}} = 2\sqrt{2}$$

that is, $x - y - 7 = \pm 4$ and $x + 7y - 31 = \pm 20$. The four combinations give the centers $(4, 1)$, $(9, 6)$, $(16, 5)$, $(11, 0)$. So there are four such rhombuses, and the inscribed circle is one of

$$\begin{aligned} (x - 4)^2 + (y - 1)^2 &= 8, & (x - 9)^2 + (y - 6)^2 &= 8, \\ (x - 16)^2 + (y - 5)^2 &= 8, & (x - 11)^2 + y^2 &= 8 \end{aligned}$$

8. Solve the equation

$$\sqrt{x^2 + x + \frac{p^2}{(x-1)^2}} = x - \frac{p}{x-1}$$

over the real numbers, where p is a real parameter. For which values of p does the equation have two different solutions?

Solution: We need $x \neq 1$, and the right-hand side must be nonnegative; under this condition squaring is an equivalent step:

$$x^2 + x + \frac{p^2}{(x-1)^2} = x^2 - \frac{2px}{x-1} + \frac{p^2}{(x-1)^2}, \quad x(x-1+2p) = 0$$

$x = 0$: the right-hand side is p , so this is a solution if $p \geq 0$.

$x = 1 - 2p$ (with $p \neq 0$, so that $x \neq 1$): the right-hand side is $1 - 2p + \frac{1}{2} = \frac{3}{2} - 2p$, so this is a solution if $p \leq \frac{3}{4}$.

The two solutions are different unless $1 - 2p = 0$. So the equation has two different solutions exactly when $0 < p \leq \frac{3}{4}$ and $p \neq \frac{1}{2}$.

Note: The solution sketch in KoMaL gives $0 < p \leq \frac{3}{4}$; for $p = \frac{1}{2}$, however, both roots are 0.