Problem Set III - 1996 (February 1996)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$x+y-\sqrt{x+y}=2, \qquad x^3+y^3=40$$

2. Prove that for every pair of real numbers (x, y),

$$x^2+y^2+1 \geq x+y+xy$$

When are the two sides equal?

3. Consider the expression $f(x)=\dfrac{x^2+2x+3}{x^2+2x+2}$.

a) Prove that $f(x)$ is defined for every real x, and its value is always positive.

b) Find the set of the possible values of $f(x)$.

c) Prove that the graph of f can be obtained by translating the graph of $g(x)=\dfrac{1}{x^2+1}$ by a vector $\mathbf{v}$. What is this vector?

4. Prove that if $\sin(\alpha-\beta)=0$, then $\sin(2\beta-\alpha)=\sin\alpha$. Is the converse of this statement true?

5. In triangle ABC, $AB=c$, $AC=b$, and $\angle BAC=\alpha$. Prove that if s_a is the length of the median to the side BC, then

$$s_a^2=\left(\frac{b+c}{2}\right)^2-bc\sin^2\frac{\alpha}{2}$$

6. The first term of a sequence of numbers is $a_1=4$, and $a_{n+1}=2a_n-1$ for every $n\geq 1$. Write down the nth term of the sequence and the sum of its first n terms as functions of n.

7. Consider the circles $(x+8)^2+(y-12)^2=100$ and $(x-4)^2+(y+4)^2=100$. Prove that the two circles are tangent to each other. Find the equation of the circle that is tangent to both circles at their point of tangency, and is also tangent to the x-axis.

8. Consider the expression $f(x)=px^2-(5p+1)x+5$, where p is a real parameter. Depending on p, when does $f(x)$ take only positive values on the open interval $(2,4)$, and when only negative values?

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