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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$x + y - \sqrt{x + y} = 2, \quad x^3 + y^3 = 40$$

Solution: With $t = \sqrt{x + y} \geq 0$, the first equation is $t^2 - t - 2 = 0$, so $t = 2$ (the root -1 is negative), and $x + y = 4$. Since $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$, the second equation gives $40 = 64 - 12xy$, so $xy = 2$. Then x and y are the roots of $t^2 - 4t + 2 = 0$:

$$(x, y) = (2 + \sqrt{2}, 2 - \sqrt{2}) \quad \text{or} \quad (2 - \sqrt{2}, 2 + \sqrt{2})$$

2. Prove that for every pair of real numbers (x, y) ,

$$x^2 + y^2 + 1 \geq x + y + xy$$

When are the two sides equal?

Solution: Twice the difference of the two sides is

$$2(x^2 + y^2 + 1 - x - y - xy) = (x - 1)^2 + (y - 1)^2 + (x - y)^2 \geq 0$$

Equality holds if and only if all three squares are 0, that is, $x = y = 1$.

3. Consider the expression $f(x) = \frac{x^2 + 2x + 3}{x^2 + 2x + 2}$.

a) Prove that $f(x)$ is defined for every real x , and its value is always positive.

b) Find the set of the possible values of $f(x)$.

c) Prove that the graph of f can be obtained by translating the graph of $g(x) = \frac{1}{x^2 + 1}$ by a vector $\mathbf{v}$. What is this vector?

$$\text{Solution: } f(x) = 1 + \frac{1}{x^2 + 2x + 2} = 1 + \frac{1}{(x + 1)^2 + 1}.$$

a) The denominator is at least 1, so f is defined everywhere, and $f(x) > 1 > 0$.

b) $0 < \frac{1}{(x + 1)^2 + 1} \leq 1$, and every value in this interval is taken, so the set of values of f is $1 < f(x) \leq 2$.

c) $f(x) = g(x + 1) + 1$, so the graph of f is the graph of g translated by $\mathbf{v} = (-1, 1)$.

4. Prove that if $\sin(\alpha - \beta) = 0$, then $\sin(2\beta - \alpha) = \sin \alpha$. Is the converse of this statement true?

Solution: If $\sin(\alpha - \beta) = 0$, then $\beta = \alpha - k\pi$ for some integer k , so $2\beta - \alpha = \alpha - 2k\pi$, and $\sin(2\beta - \alpha) = \sin \alpha$.

The converse is not true: for example, for $\alpha = 0$ and $\beta = \frac{\pi}{2}$, $\sin(2\beta - \alpha) = \sin \pi = 0 = \sin \alpha$, but $\sin(\alpha - \beta) = -1$.
(In fact $\sin(2\beta - \alpha) - \sin \alpha = -2 \cos \beta \sin(\alpha - \beta)$, so the converse fails exactly when $\cos \beta = 0$.)

5. In triangle ABC , $AB = c$, $AC = b$, and $\angle BAC = \alpha$. Prove that if s_a is the length of the median to the side BC , then

$$s_a^2 = \left(\frac{b+c}{2}\right)^2 - bc \sin^2 \frac{\alpha}{2}$$

Solution: Let A_1 be the midpoint of BC and $\varphi = \angle CA_1A$; then $\angle BA_1A = 180^\circ - \varphi$. By the Law of Cosines in the triangles AA_1C and AA_1B ,

$$b^2 = \frac{a^2}{4} + s_a^2 - as_a \cos \varphi, \quad c^2 = \frac{a^2}{4} + s_a^2 + as_a \cos \varphi$$

Adding: $b^2 + c^2 = 2s_a^2 + \frac{a^2}{2}$. With $a^2 = b^2 + c^2 - 2bc \cos \alpha$:

$$4s_a^2 = 2(b^2 + c^2) - a^2 = b^2 + c^2 + 2bc \cos \alpha = (b+c)^2 - 2bc(1 - \cos \alpha)$$

and since $1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}$, dividing by 4 gives the statement.

6. The first term of a sequence of numbers is $a_1 = 4$, and $a_{n+1} = 2a_n - 1$ for every $n \geq 1$. Write down the n th term of the sequence and the sum of its first n terms as functions of n .

Solution: $a_{n+1} - 1 = 2(a_n - 1)$, so the numbers $a_n - 1$ form a geometric sequence with first term 3 and ratio 2:
 $a_n - 1 = 3 \cdot 2^{n-1}$. So

$$a_n = 3 \cdot 2^{n-1} + 1, \quad S_n = 3(1 + 2 + \cdots + 2^{n-1}) + n = 3 \cdot 2^n + n - 3$$

7. Consider the circles $(x+8)^2 + (y-12)^2 = 100$ and $(x-4)^2 + (y+4)^2 = 100$. Prove that the two circles are tangent to each other. Find the equation of the circle that is tangent to both circles at their point of tangency, and is also tangent to the x -axis.

Solution: Both radii are 10, and the distance of the centers $C_1(-8, 12)$ and $C_2(4, -4)$ is $\sqrt{144 + 256} = 20$, so the circles touch each other at the midpoint $E(-2, 4)$ of C_1C_2 .

A circle tangent to both circles at E has its center $C(u, v)$ on the line C_1C_2 , that is, $4u + 3v = 4$, and if it is tangent to the x -axis, its radius is $|v|$. E is on the circle:

$$(-2-u)^2 + (4-v)^2 = v^2$$

With $u = 1 - \frac{3}{4}v$: $\left(\frac{3}{4}v - 3\right)^2 + (4-v)^2 = v^2$, that is, $9v^2 - 200v + 400 = 0$, so $v = 20$ or $v = \frac{20}{9}$. There are two such circles:

$$(x+14)^2 + (y-20)^2 = 400 \quad \text{and} \quad \left(x + \frac{2}{3}\right)^2 + \left(y - \frac{20}{9}\right)^2 = \frac{400}{81}$$

8. Consider the expression $f(x) = px^2 - (5p + 1)x + 5$, where p is a real parameter. Depending on p , when does $f(x)$ take only positive values on the open interval $(2, 4)$, and when only negative values?

Solution: $f(x) = (px - 1)(x - 5)$, and on $(2, 4)$ the factor $x - 5$ is negative. So $f(x) > 0$ on the whole interval if and only if $px < 1$ for every x in $(2, 4)$, that is, $p \leq \frac{1}{4}$ (for $p \leq 0$ this is clear, for $p > 0$ we need $4p \leq 1$). And $f(x) < 0$ on the whole interval if and only if $px > 1$ there, that is, $2p \geq 1$, $p \geq \frac{1}{2}$.

So f takes only positive values on $(2, 4)$ for $p \leq \frac{1}{4}$, only negative values for $p \geq \frac{1}{2}$, and both positive and negative values for $\frac{1}{4} < p < \frac{1}{2}$.