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Pdf file: KoMaL, March 1996, page 144

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Let $a = \sin^2 1995^\circ$ and $b = \cos^2 1995^\circ$. Find the exact value of $\log_2 (ab)^{-\frac{1}{2}}$.

Solution: $1995^\circ = 5 \cdot 360^\circ + 195^\circ$, and $\sin^2 195^\circ = \sin^2 15^\circ$, $\cos^2 195^\circ = \cos^2 15^\circ$. So

$$4ab = (2 \sin 15^\circ \cos 15^\circ)^2 = \sin^2 30^\circ = \frac{1}{4}, \quad ab = \frac{1}{16}$$

$$\text{and } \log_2 \left(\frac{1}{16} \right)^{-\frac{1}{2}} = \log_2 4 = 2.$$

2. A cube and a regular tetrahedron have the same surface area. Find the ratio of their volumes.

Solution: Let the edge of the cube be a and the edge of the tetrahedron b . The surface areas are $6a^2$ and $\sqrt{3}b^2$, so $\frac{a^2}{b^2} = \frac{\sqrt{3}}{6}$. The volume of the regular tetrahedron is $\frac{\sqrt{2}}{12}b^3$, so

$$\frac{V_{\text{cube}}}{V_{\text{tetrahedron}}} = \frac{12}{\sqrt{2}} \left(\frac{a}{b} \right)^3 = \frac{12}{\sqrt{2}} \left(\frac{\sqrt{3}}{6} \right)^{\frac{3}{2}} = \sqrt[4]{3} \approx 1.316$$

3. The shortest altitude of an obtuse triangle is equal to the perpendicular projection of the shortest side onto the longest side. Express the radius of the circumscribed circle in terms of the sides of the triangle.

Solution: Let the obtuse angle be at C , so $AB = c$ is the longest side, and let $BC = a$ be the shortest side and $AC = b$. The shortest altitude is the one to the longest side, CT (with T on AB), and the projection of BC onto AB is BT . So $CT = BT$, the right triangle CTB is isosceles, and the angle at B is 45° . By the Law of Sines,

$$r = \frac{b}{2 \sin 45^\circ} = \frac{b}{\sqrt{2}}$$

(Equivalently, since $b^2 = a^2 + c^2 - \sqrt{2}ac$ by the Law of Cosines, $r = \sqrt{\frac{a^2 + c^2 - \sqrt{2}ac}{2}}$.)

4. For the functions $f(x) = x^2 - 2x + 3$ and $g(x) = x^2 - 4x + 7$, defined on the real numbers, find all positive integers n for which $f(n)$ is the geometric mean of n and $g(n)$.

Solution: $f(n) > 0$ and $g(n) > 0$, so the condition is $f(n)^2 = n \cdot g(n)$:

$$(n^2 - 2n + 3)^2 = n(n^2 - 4n + 7), \quad n^4 - 5n^3 + 14n^2 - 19n + 9 = 0$$

An integer root must divide 9, and a positive one can only be 1, 3, or 9. Substituting, only $n = 1$ works: $f(1) = 2$, $g(1) = 4$, and $2 = \sqrt{1 \cdot 4}$.

5. The real numbers a, b, c, d are not less than $\frac{1}{3}$, and $a + b + c + d = 2$. Prove that

$$\sqrt{6a-2} + \sqrt{6b-2} + \sqrt{6c-2} + \sqrt{6d-2} \leq 4$$

Solution: The square roots are defined, since $6a - 2 \geq 0$, etc. By the inequality between the arithmetic and quadratic means of four numbers,

$$\frac{\sqrt{6a-2} + \sqrt{6b-2} + \sqrt{6c-2} + \sqrt{6d-2}}{4} \leq \sqrt{\frac{(6a-2) + (6b-2) + (6c-2) + (6d-2)}{4}} = \sqrt{\frac{6 \cdot 2 - 8}{4}} = 1$$

Multiplying by 4 gives the statement. Equality holds for $a = b = c = d = \frac{1}{2}$.

6. The vertices of a triangle are $A(5, -2)$, $B(-3, 2)$, $C(-2, -1)$. Find the equation of its circumscribed circle.

Solution: The center is the intersection of the perpendicular bisectors. The perpendicular bisector of BC has normal vector $\overrightarrow{BC} = (1, -3)$ and goes through $\left(-\frac{5}{2}, \frac{1}{2}\right)$: $x - 3y = -4$. The perpendicular bisector of CA has normal vector $\overrightarrow{CA} = (7, -1)$ and goes through $\left(\frac{3}{2}, -\frac{3}{2}\right)$: $7x - y = 12$. They meet at $K(2, 2)$, and $KA = KB = KC = 5$. The circle is

$$(x - 2)^2 + (y - 2)^2 = 25$$

(This triangle has an obtuse angle at C and a 45° angle at B , so we could also use $r = \frac{CA}{2 \sin 45^\circ} = \frac{\sqrt{50}}{\sqrt{2}} = 5$.)

7. Six different real numbers are six consecutive terms of an arithmetic sequence. The first, the second, and the sixth terms are three consecutive terms of a geometric sequence. Find the ratio of the first term and the difference of the arithmetic sequence. Which term of the arithmetic sequence is the tenth term of the geometric sequence? Show that the arithmetic sequence contains every term of the geometric sequence.

Solution: Let the numbers be $a, a + d, \dots, a + 5d$, with $d \neq 0$. The condition is $(a + d)^2 = a(a + 5d)$, that is, $d^2 = 3ad$, so $d = 3a$ and $\frac{a}{d} = \frac{1}{3}$ (and $a \neq 0$). Then the geometric sequence is $a, 4a, 16a, \dots$, with ratio 4.

Its tenth term is $a \cdot 4^9 = 262144a$. The k th term of the arithmetic sequence is $a + (k - 1)3a$, so $1 + 3(k - 1) = 262144$, $k = 87382$: it is the 87382nd term.

In general, the term $a \cdot 4^n$ of the geometric sequence is $a + \frac{4^n - 1}{3} \cdot 3a = a + \frac{4^n - 1}{3} \cdot d$, and $4^n - 1$ is divisible by $4 - 1 = 3$, so it is term number $\frac{4^n - 1}{3} + 1$ of the arithmetic sequence.

8. Solve the following equation over the real numbers.

$$\tan^2 x + \tan^2 x \tan 2x \tan 3x - \tan 3x + \tan 2x - 6 \tan 3x \tan 2x - 6 = 0$$

Solution: We need $\cos x$, $\cos 2x$, $\cos 3x$ to be nonzero. Grouping the terms:

$$\tan^2 x (1 + \tan 2x \tan 3x) - 6 (1 + \tan 2x \tan 3x) - (\tan 3x - \tan 2x) = 0$$

Here $1 + \tan 2x \tan 3x = \frac{\cos 3x \cos 2x + \sin 3x \sin 2x}{\cos 2x \cos 3x} = \frac{\cos x}{\cos 2x \cos 3x} \neq 0$, so by the subtraction formula

$$\tan x = \tan(3x - 2x) = \frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x}, \quad \tan 3x - \tan 2x = \tan x (1 + \tan 2x \tan 3x)$$

So

$$(\tan^2 x - \tan x - 6)(1 + \tan 2x \tan 3x) = 0$$

The second factor is not 0, so $\tan x = 3$ or $\tan x = -2$ (and then $\tan 2x$ and $\tan 3x$ are defined). The solutions are

$$x = \tan^{-1} 3 + k\pi \approx 71.57^\circ + k \cdot 180^\circ \quad \text{or} \quad x = -\tan^{-1} 2 + k\pi \approx -63.43^\circ + k \cdot 180^\circ$$