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In memory of Paula Soós, teacher

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sum of the first six terms of an arithmetic sequence $(a_1, a_2, \dots, a_n, \dots)$ is 42, and $a_2 a_5 = 13$. Find the first term and the difference of the sequence.

Solution: Write the first six terms as $a - 5t, a - 3t, a - t, a + t, a + 3t, a + 5t$ (the difference is $d = 2t$). Then $6a = 42$, so $a = 7$, and $(a - 3t)(a + 3t) = 49 - 9t^2 = 13$, so $t = \pm 2$.

If $t = 2$: $d = 4$ and $a_1 = -3$ (the sequence $-3, 1, 5, 9, 13, 17, \dots$).

If $t = -2$: $d = -4$ and $a_1 = 17$.

2. The point C divides the segment AB in the ratio $1 : 2$, that is, $2AC = CB$. We draw semicircles over the segments AB , AC , and CB as diameters, on the same side of the line AB . The radius of the circle tangent to these three semicircles is $r = 12$ units. Find the length of the segment AB .

Solution: Let $AB = 12x$, so $AC = 4x$ and $CB = 8x$. Let F, O_1, O_2 be the midpoints of AB, AC, CB (the centers of the semicircles, with radii $6x, 2x, 4x$), and let O_3 be the center of the circle. The circle touches the big semicircle from the inside and the small ones from the outside, so

$$O_1 O_3 = 2x + 12, \quad O_2 O_3 = 4x + 12, \quad F O_3 = 6x - 12, \quad \text{and} \quad O_1 F = 4x, \quad F O_2 = 2x$$

Let $\alpha = \angle O_1 F O_3$; then $\angle O_2 F O_3 = 180^\circ - \alpha$. By the Law of Cosines in the triangles $O_1 F O_3$ and $O_2 F O_3$:

$$\cos \alpha = \frac{(4x)^2 + (6x - 12)^2 - (2x + 12)^2}{2 \cdot 4x \cdot (6x - 12)} = \frac{x - 4}{x - 2}, \quad -\cos \alpha = \frac{(2x)^2 + (6x - 12)^2 - (4x + 12)^2}{2 \cdot 2x \cdot (6x - 12)} = \frac{x - 10}{x - 2}$$

Adding: $(x - 4) + (x - 10) = 0$, so $x = 7$, and $AB = 84$ units.

3. Solve the following system of equations over the pairs of real numbers.

$$x = \frac{y^2 + 2}{2y + 1}, \quad y = \frac{x^2 + 2}{2x + 1}$$

Solution: We need $x \neq -\frac{1}{2}, y \neq -\frac{1}{2}$. The equations are $2xy + x = y^2 + 2$ and $2xy + y = x^2 + 2$. Subtracting: $x - y = y^2 - x^2$, that is, $(x - y)(1 + x + y) = 0$.

If $x = y$: $x^2 + x - 2 = 0$, so $x = y = 1$ or $x = y = -2$.

If $x + y = -1$: substituting $y = -1 - x$ into the second equation gives $3x^2 + 3x + 3 = 0$, which has no real solution.

The solutions are $(1, 1)$ and $(-2, -2)$.

4. The area of a triangle is $T = \frac{1}{4\sqrt{3}}(c^2 - a^2 - b^2)$, where a , b , and c are the lengths of its sides. Find the angle opposite the side c .

Solution: $T = \frac{1}{2}ab \sin \gamma$, and by the Law of Cosines $c^2 - a^2 - b^2 = -2ab \cos \gamma$. So

$$\frac{1}{2}ab \sin \gamma = -\frac{2ab \cos \gamma}{4\sqrt{3}}, \quad \tan \gamma = -\frac{1}{\sqrt{3}}, \quad \gamma = 150^\circ$$

5. Consider the expression $f(x) = \frac{11}{2} - \frac{1}{2} \cos 4x - \frac{4 \tan x}{1 + \tan^2 x}$.

a) Find the largest set D on which the expression is defined.

b) Find the largest and the smallest values of the function $x \mapsto f(x)$ defined on D . Where does the function take these values?

Solution: a) $\tan x$ must be defined: D is the set of all real numbers except $\frac{\pi}{2} + k\pi$.

b) On D , $\frac{4 \tan x}{1 + \tan^2 x} = 4 \sin x \cos x = 2 \sin 2x$, and $\cos 4x = 1 - 2 \sin^2 2x$. So

$$f(x) = 5 + \sin^2 2x - 2 \sin 2x = (\sin 2x - 1)^2 + 4$$

The largest value is 8, where $\sin 2x = -1$: $x = \frac{3\pi}{4} + k\pi$. The smallest value is 4, where $\sin 2x = 1$: $x = \frac{\pi}{4} + k\pi$. (These points are in D .)

6. We deposited 100 000 Ft in a bank at compound interest. In the first year the bank paid 24% interest, in the second year it reduced the interest rate by p percent (percentage points), and in the third year again by p percent. So at the end of the third year they paid back 18 054.4 Ft less than we hoped for at the 24% rate. Find the value of p .

Solution: At 24% we would have had $100\,000 \cdot 1.24^3 = 190\,662.4$ Ft, so we got 172 608 Ft. So

$$100\,000 \cdot 1.24 \cdot \left(1 + \frac{24-p}{100}\right) \left(1 + \frac{24-2p}{100}\right) = 172\,608, \quad p^2 - 186p + 728 = 0$$

so $p = 4$ or $p = 182$. The rates must stay positive, so $p = 4$ (the rates were 24%, 20%, 16%).

7. Find the equation of the circle that is tangent to the line $x = 12$ at its point with second coordinate 8, and is also tangent to the circle $x^2 + y^2 = 16$.

Solution: The center is on the line $y = 8$, say $(u, 8)$, and the radius is $|12 - u|$. The circle is tangent to the circle with center O and radius 4 if the distance of the centers is the sum or the difference of the radii: $\sqrt{u^2 + 64} = |12 - u| \pm 4$. Squaring (for $u < 12$): $u^2 + 64 = (12 - u)^2 \pm 8(12 - u) + 16$, which gives $u = 6$ (with $+$) or $u = 0$ (with $-$). (For $u > 12$ there is no solution.) There are two such circles:

$$x^2 + (y - 8)^2 = 144 \quad \text{and} \quad (x - 6)^2 + (y - 8)^2 = 36$$

(the first one contains the given circle, the second one is outside it).

8. For which values of the real parameter p does the equation

$$2 - \frac{x}{x+2p} = \frac{2}{x-2p} + \frac{x^2+2x-6p^2}{x^2-4p^2}$$

have a solution? For which values of p is the root of the equation less than $2p$?

Solution: We need $x \neq \pm 2p$. Multiplying by $x^2 - 4p^2$ and simplifying, the equation becomes

$$(p-2)x = p^2 + 2p$$

If $p = 2$, there is no solution. Otherwise $x = \frac{p^2+2p}{p-2}$, and this is a solution unless it equals $2p$ or $-2p$: $\frac{p^2+2p}{p-2} = 2p$ gives $p = 0$ or $p = 6$, and $\frac{p^2+2p}{p-2} = -2p$ gives $p = 0$ or $p = \frac{2}{3}$. So the equation has a solution (exactly one) if and only if $p \neq 0, \frac{2}{3}, 2, 6$.

The root is less than $2p$ if $\frac{p^2+2p}{p-2} - 2p = \frac{-p(p-6)}{p-2} < 0$, that is, $\frac{p(p-6)}{p-2} > 0$: $0 < p < 2$ or $p > 6$, and excluding $p = \frac{2}{3}$: $0 < p < \frac{2}{3}, \frac{2}{3} < p < 2$, or $p > 6$.