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In memory of Jenő Berkes, teacher

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Prove that in every triangle

$$a(\sin \beta - \sin \gamma) + b(\sin \gamma - \sin \alpha) + c(\sin \alpha - \sin \beta) = 0$$

Solution: By the Law of Sines $a \sin \beta = b \sin \alpha$, $a \sin \gamma = c \sin \alpha$, and $b \sin \gamma = c \sin \beta$. So

$$(a \sin \beta - b \sin \alpha) + (b \sin \gamma - c \sin \beta) + (c \sin \alpha - a \sin \gamma) = 0 + 0 + 0 = 0$$

and the left-hand side is exactly the given expression, regrouped.

2. Find the smallest positive root and the negative root with the smallest absolute value of the equation

$$\cos \pi (x^2 - 2x + 1) + \cos \pi x^2 = 0$$

Solution: Since $-\cos \varphi = \cos(\varphi + \pi)$, the equation is $\cos \pi (x - 1)^2 = \cos(\pi x^2 + \pi)$, so

$$\pi (x - 1)^2 = \pi (x^2 + 1) + 2k\pi \quad \text{or} \quad \pi (x - 1)^2 = -\pi (x^2 + 1) + 2n\pi$$

for some integers k, n . The first case gives $x = -k$, so every integer is a root. The second case gives $x^2 - x + 1 - n = 0$, that is, $x = \frac{1 \pm \sqrt{4n - 3}}{2}$ ($n \geq 1$). The roots of the second kind are either at least 1 or at most 0, and the negative one closest to 0 is $\frac{1 - \sqrt{5}}{2}$ (for $n = 2$). So the smallest positive root is 1, and the negative root with the smallest absolute value is $\frac{1 - \sqrt{5}}{2} \approx -0.618$.

3. Solve the equation

$$\sqrt{x + 2\sqrt{x+2} + 3} + \sqrt{x - 2\sqrt{x+2} + 3} = a$$

for $a = 1$, for $a = 2$, for $a = 4$, and for $a = 2\sqrt{x+2}$.

Solution: We need $x \geq -2$. Let $u = \sqrt{x+2} \geq 0$; then $x + 3 \pm 2\sqrt{x+2} = u^2 \pm 2u + 1 = (u \pm 1)^2$, so the left-hand side is

$$|u+1| + |u-1| = \begin{cases} 2 & \text{if } 0 \leq u \leq 1, \text{ that is, } -2 \leq x \leq -1 \\ 2u = 2\sqrt{x+2} & \text{if } u \geq 1, \text{ that is, } x \geq -1 \end{cases}$$

$a = 1$: no solution (the left-hand side is at least 2).

$a = 2$: every x with $-2 \leq x \leq -1$.

$a = 4$: $2\sqrt{x+2} = 4$, so $x = 2$.

$a = 2\sqrt{x+2}$: every $x \geq -1$ (for $-2 \leq x < -1$ we would need $2 = 2\sqrt{x+2}$, which fails).

4. For two years (twenty-four months), at the beginning of every month we deposit 5000 forints in a bank at 2% monthly interest. At the end of the second year (the twenty-fourth month) for the first time, and then at the end of every month, we withdraw the same amount, so that we withdraw the last installment at the end of the third year (the thirty-sixth month), and then no money is left in the bank. How much money will we have in the bank at the end of the second year, before and after the withdrawal?

Solution: At the end of the 24th month, before the withdrawal, we have

$$T = 5000 (1.02^{24} + 1.02^{23} + \cdots + 1.02) = 5000 \cdot 1.02 \cdot \frac{1.02^{24} - 1}{1.02 - 1} \approx 155\,151 \text{ Ft}$$

There are 13 withdrawals of x forints (at the end of months 24, 25, ..., 36). At the end of month 36 the money would be worth $T \cdot 1.02^{12}$, and the withdrawals $x (1.02^{12} + \cdots + 1) = x \cdot \frac{1.02^{13} - 1}{0.02}$. These are equal, so

$$x = \frac{T \cdot 1.02^{12} \cdot 0.02}{1.02^{13} - 1} \approx 13\,404 \text{ Ft}$$

So after the first withdrawal we have about $155\,151 - 13\,404 = 141\,747$ Ft in the bank.

5. Let a and b be real numbers such that $a < 2b$ and $ab = 1$. Prove that

$$\frac{a^2 + 4b^2}{a - 2b} + 4 \leq 0$$

Solution: Since $2b - a > 0$, the statement is equivalent to $\frac{a^2 + 4b^2}{2b - a} \geq 4$. Using $ab = 1$, $a^2 + 4b^2 = (2b - a)^2 + 4ab = (2b - a)^2 + 4$, so

$$\frac{a^2 + 4b^2}{2b - a} = (2b - a) + \frac{4}{2b - a} \geq 2\sqrt{4} = 4$$

by the inequality between the arithmetic and geometric means.

6. The point $P(1, 1)$ and the lines $e: x + y = -1$ and $f: 8x + 3y = 7$ are given. Find the equation of the line through P whose segment between e and f is divided by P in the ratio $3 : 2$, counting from the line e .

Solution: Let the endpoints be $E(u, -u - 1)$ on e and $F\left(3t, \frac{7}{3} - 8t\right)$ on f . Since $EP : PF = 3 : 2$, $P = \frac{2E + 3F}{5}$:

$$5 = 2u + 9t, \quad 5 = -2u - 2 + 7 - 24t$$

Adding: $10 = 5 - 15t$, so $t = -\frac{1}{3}$ and $u = 4$. So $E(4, -5)$, $F(-1, 5)$, and the line is

$$2x + y = 3$$

7. Three times one root of the equation $x^2 - px + q = 0$ is a root of the equation $x^2 + (p + 4)x - 3q = 0$. Find p and q , if $p^2 - 4q = 9$.

Solution: Let x_1, x_2 be the roots of the first equation, and $3x_1, x_3$ the roots of the second one. By Viète's formulas

$$x_1 + x_2 = p, \quad x_1 x_2 = q, \quad 3x_1 + x_3 = -p - 4, \quad 3x_1 x_3 = -3q$$

From the products, $x_1(x_2 + x_3) = 0$, and from the sums, $3x_2 - x_3 = 4p + 4$.

If $x_1 = 0$: $q = 0$, and $p^2 = 9$, so $p = \pm 3$.

If $x_3 = -x_2$: $4x_2 = 4p + 4$, so $x_2 = p + 1$ is a root of the first equation: $(p + 1)^2 - p(p + 1) + q = 0$, so $q = -p - 1$, and $p^2 + 4p + 4 = 9$ gives $p = 1, q = -2$ or $p = -5, q = 4$.

The solutions are $(p, q) = (3, 0), (-3, 0), (1, -2), (-5, 4)$. (For example, for $p = 1, q = -2$ the roots of the first equation are 2 and -1 , and -3 is a root of $x^2 + 5x + 6 = 0$.)

8. The real numbers x, y , and z satisfy the following conditions: $x + y = 2z + 1$ and $x^2 + y^2 = z^2 + 4z$. For which value of z is the product xy the smallest, and for which the largest? What are the smallest and the largest values of xy ?

Solution: $xy = \frac{(x + y)^2 - (x^2 + y^2)}{2} = \frac{(2z + 1)^2 - z^2 - 4z}{2} = \frac{3z^2 + 1}{2}$. The numbers x and y are the roots of $t^2 - (2z + 1)t + \frac{3z^2 + 1}{2} = 0$, which are real if and only if

$$(2z + 1)^2 - 2(3z^2 + 1) = 1 - 2(z - 1)^2 \geq 0, \quad 1 - \frac{\sqrt{2}}{2} \leq z \leq 1 + \frac{\sqrt{2}}{2}$$

On this interval $z > 0$, so $\frac{3z^2 + 1}{2}$ is increasing. So xy is smallest for $z = 1 - \frac{\sqrt{2}}{2}$, with value $\frac{11 - 6\sqrt{2}}{4} \approx 0.63$, and largest for $z = 1 + \frac{\sqrt{2}}{2}$, with value $\frac{11 + 6\sqrt{2}}{4} \approx 4.87$.