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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. a) Solve the following equation over the natural numbers.

$$\log_2 3 + \log_{2^2} 3^2 + \cdots + \log_{2^n} 3^n = \log_2 81$$

- b) Find the real solutions of the following equation.

$$2\sqrt{3^{2x-2} + 9^x - 81} = 96 - 9^{x-1} - 3^{2x}$$

Solution: a) $\log_{2^k} 3^k = \frac{k \log_2 3}{k} = \log_2 3$, so the left-hand side is $n \log_2 3$, and the right-hand side is $4 \log_2 3$. So $n = 4$.

b) With $a = 10 \cdot 9^{x-1}$: $3^{2x-2} + 9^x = 9^{x-1} + 9 \cdot 9^{x-1} = a$ and $9^{x-1} + 3^{2x} = a$, so the equation is $2\sqrt{a - 81} = 96 - a$, which requires $81 \leq a \leq 96$. Squaring:

$$4a - 324 = 9216 - 192a + a^2, \quad a^2 - 196a + 9540 = 0, \quad a = 90 \text{ or } a = 106$$

Only $a = 90$ is allowed, so $9^{x-1} = 9$, and $x = 2$.

2. The legs a and b of a right triangle satisfy

$$2 \cdot \log \frac{a+b}{4} = \log a + \log b$$

Find the value of $\sin 2\alpha$, where α is the angle opposite a .

Solution: The condition means $\frac{(a+b)^2}{16} = ab$, that is, $a^2 + b^2 = 14ab$. With c the hypotenuse, $c^2 = 14ab$, so

$$\sin \alpha \cos \alpha = \frac{a}{c} \cdot \frac{b}{c} = \frac{1}{14}, \text{ and}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{1}{7}$$

3. Prove that if the perpendicular bisector of a diagonal of a rectangle divides the longer side a of the rectangle into two parts, one of which is equal to the shorter side b , then $a + b$ is the hypotenuse of an isosceles right triangle with legs a .

Solution: Let the rectangle be $ABCD$ with $AB = CD = a$ and $BC = DA = b$, and let the perpendicular bisector of the diagonal AC meet CD at G . Every point of the perpendicular bisector is equally far from A and C , so $GA = GC$. The part GC cannot be equal to b : then $GA = b = DA$, but GA is the hypotenuse of the right triangle ADG , so $GA > DA$. So $DG = b$ and $GA = GC = a - b$. By the Pythagorean theorem in triangle ADG :

$$b^2 + b^2 = (a - b)^2, \quad b^2 = a^2 - 2ab, \quad (a + b)^2 = a^2 + 2ab + b^2 = 2a^2$$

So $a + b = \sqrt{a^2 + a^2}$ is the hypotenuse of the isosceles right triangle with legs a .

4. In a triangle (with the usual notation) the angle γ is opposite the side c , and m_c is the altitude to the side c . Prove that

$$\frac{c}{\sin \gamma} > m_c$$

Solution: Computing the area in two ways, $ab \sin \gamma = c m_c$, so $\frac{c}{\sin \gamma} = \frac{ab}{m_c}$. The altitude m_c is at most the sides a and b (the perpendicular is the shortest segment from C to the line AB), and it cannot be equal to both (then both CA and CB would be perpendicular to AB). So $m_c^2 < ab$, that is, $\frac{ab}{m_c} > m_c$.

5. The vertices of the broken line $A_0 A_1 \dots A_n$ are given as follows:

$$A_{4k} (0, -2k), \quad A_{4k+1} (4k+1, 2k+1), \quad A_{4k+2} (-1, 2k+1), \quad A_{4k+3} (-4k-4, -2k-2)$$

where k is a nonnegative integer. How long is this broken line from A_0 to A_{1997} ?

Solution: By the Pythagorean theorem,

$$A_{4k} A_{4k+1} = (4k+1) \sqrt{2}, \quad A_{4k+1} A_{4k+2} = 4k+2, \quad A_{4k+2} A_{4k+3} = (4k+3) \sqrt{2}, \quad A_{4k+3} A_{4k+4} = 4k+4$$

So the segment $A_j A_{j+1}$ has length $(j+1) \sqrt{2}$ if j is even and $j+1$ if j is odd. From A_0 to A_{1997} there are 1997 segments, so the length is

$$(1 + 3 + \dots + 1997) \sqrt{2} + (2 + 4 + \dots + 1996) = 999^2 \sqrt{2} + 998 \cdot 999 = 998\,001 \sqrt{2} + 997\,002$$

6. The opposite edges of a tetrahedron are pairwise equal, and the three different edge lengths are $2\sqrt{85}$, $3\sqrt{41}$, $\sqrt{421}$ units. Find the distances between the pairs of opposite (skew) edges.

Solution: Such a tetrahedron can be placed in a rectangular box so that its edges are diagonals of the faces of the box (two opposite edges are the two diagonals of two opposite faces). Let the edges of the box be a , b , c . Then

$$a^2 + b^2 = 340, \quad b^2 + c^2 = 369, \quad c^2 + a^2 = 421$$

so $a^2 = 196$, $b^2 = 144$, $c^2 = 225$. The distance of two opposite edges is the distance of the two opposite faces containing them, that is, the edge of the box perpendicular to those faces. So the distances are 15 (between the edges of length $2\sqrt{85}$), 14 (between the edges of length $3\sqrt{41}$), and 12 (between the edges of length $\sqrt{421}$).

7. Three points A, B, C form a right triangle on the map, with the right angle at B . The difference in elevation between A and B is 80 meters, and between B and C it is 120 meters. A straight road with a 10° upward slope leads from A to B , and a straight road with a 12° upward slope leads from B to C . Find the straight-line distance between A and C . What is the angle ABC in reality?

Solution: The horizontal distances (on the map) are $80 \cot 10^\circ \approx 453.70$ m from A to B and $120 \cot 12^\circ \approx 564.56$ m from B to C , and these are perpendicular. C is 200 m higher than A . So

$$AC = \sqrt{453.70^2 + 564.56^2 + 200^2} \approx 751.4 \text{ m}$$

The roads are $AB = \frac{80}{\sin 10^\circ} \approx 460.70$ m and $BC = \frac{120}{\sin 12^\circ} \approx 577.17$ m long, and by the Law of Cosines

$$\cos \angle ABC = \frac{AB^2 + BC^2 - AC^2}{2 \cdot AB \cdot BC} \approx -0.0361, \quad \angle ABC \approx 92.07^\circ$$

8. Take the sum of the cubes of the first n positive integers and divide it by the sum of the squares of the first n positive integers. Can the quotient be an integer, if $n > 1$ is an integer?

Solution: No. Using $1^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ and $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, the quotient is

$$\frac{3n(n+1)}{2(2n+1)}$$

Since $2n+1$ is odd and relatively prime to n and to $n+1$ (because $2n+1 = 2 \cdot n + 1 = 2(n+1) - 1$), it can divide $3n(n+1)$ only if it divides 3, that is, $2n+1 = 3$, $n = 1$. So for $n > 1$ the quotient is never an integer.