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In memory of István Bóka, teacher (István Bóka was the author's mathematics teacher in 1941–44 at the Baross Gábor High School in Szeged.)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

$$\text{a) } 9^x - 2^{x+\frac{1}{2}} = 2^{x+\frac{7}{2}} - 3^{2x-1} \quad \text{b) } 9^x - 2^{x+\frac{1}{2}} = 2^{x+\frac{3}{2}} - 3^{2x-1}$$

Solution: Since $3^{2x-1} = \frac{9^x}{3}$, both equations have $9^x + \frac{9^x}{3} = \frac{4}{3} \cdot 9^x$ on the left after rearranging.

$$\text{a) } \frac{4}{3} \cdot 9^x = 2^x (\sqrt{2} + 8\sqrt{2}) = 9\sqrt{2} \cdot 2^x, \text{ so } \left(\frac{9}{2}\right)^x = \frac{27\sqrt{2}}{4} = \left(\frac{9}{2}\right)^{\frac{3}{2}}, \text{ and } x = \frac{3}{2}.$$

$$\text{b) } \frac{4}{3} \cdot 9^x = 2^x (\sqrt{2} + 2\sqrt{2}) = 3\sqrt{2} \cdot 2^x, \text{ so } \left(\frac{9}{2}\right)^x = \frac{9\sqrt{2}}{4} = \frac{9}{2\sqrt{2}}, \text{ and } x = \log_{\frac{9}{2}} \frac{9}{2\sqrt{2}} \approx 0.770.$$

2. The terms a_1, a_2, a_4, a_5 of an arithmetic sequence satisfy $a_1 a_2 = 3$ and $a_4 a_5 = 63$. Find the first term and the difference of the sequence.

Solution: Let a be the third term and d the difference. Then $(a - 2d)(a - d) = 3$ and $(a + 2d)(a + d) = 63$, that is, $a^2 - 3ad + 2d^2 = 3$ and $a^2 + 3ad + 2d^2 = 63$. Subtracting and adding: $ad = 10$ and $a^2 + 2d^2 = 33$. Substituting $d = \frac{10}{a}$:

$$a^4 - 33a^2 + 200 = 0, \quad a^2 = 25 \text{ or } a^2 = 8$$

So there are four such sequences:

$$a_1 = 1, d = 2; \quad a_1 = -1, d = -2; \quad a_1 = -3\sqrt{2}, d = \frac{5\sqrt{2}}{2}; \quad a_1 = 3\sqrt{2}, d = -\frac{5\sqrt{2}}{2}$$

3. Prove that if $\sin(\alpha - \beta) = 0$, then $\sin(2\beta - \alpha) = \sin \alpha$.

Solution: If $\sin(\alpha - \beta) = 0$, then $\beta = \alpha - k\pi$ for some integer k , so $2\beta - \alpha = \alpha - 2k\pi$, and $\sin(2\beta - \alpha) = \sin \alpha$.

4. Find the values of the parameter m for which one root of the equation

$$mx^2 + 2(m-1)x - 4 = 0$$

is less than 3, and the other root is greater than 3.

Solution: We need $m \neq 0$ (for two roots). The equation factors as $(x+2)(mx-2) = 0$, so the roots are -2 and $\frac{2}{m}$.

Since $-2 < 3$, we need $\frac{2}{m} > 3$, that is, $0 < m < \frac{2}{3}$.

5. Find the legs of the right triangle whose hypotenuse is $4\sqrt{10}$ units, and whose angle bisector from the right angle is $3\sqrt{2}$ units long.

Solution: Let the legs be a and b . The bisector f divides the triangle into two triangles with 45° angles at the right-angle vertex, so (adding their areas) $\frac{ab}{2} = \frac{(a+b)f \sin 45^\circ}{2}$, that is, $f = \frac{\sqrt{2}ab}{a+b}$. So $3(a+b) = ab$, and $a^2 + b^2 = 160$ gives

$$(a+b)^2 - 6(a+b) - 160 = 0, \quad a+b = 16, \quad ab = 48$$

so the legs are 4 and 12 units long.

6. The circles k_1 and k_2 have equal radii, and their common points are $A(3, 2)$ and $B(7, 6)$. Find the equations of the circles, if the center of one of them is on the line $4x + 5y = 20$.

Solution: Both centers are on the perpendicular bisector of AB , which goes through the midpoint $F(5, 4)$ with normal vector $(1, 1)$: $x + y = 9$. Its intersection with $4x + 5y = 20$ is $C_1(25, -16)$. The circles have equal radii, so the other center is the reflection of C_1 in F : $C_2(-15, 24)$. The radius squared is $C_1A^2 = 22^2 + 18^2 = 808$. The circles are

$$(x-25)^2 + (y+16)^2 = 808 \quad \text{and} \quad (x+15)^2 + (y-24)^2 = 808$$

7. Solve the following equation over the pairs of real numbers.

$$x^2 + 4x \cos xy + 4 = 0$$

Solution: The equation can be written as

$$(x + 2 \cos xy)^2 + 4 \sin^2 xy = 0$$

so $\sin xy = 0$ and $x = -2 \cos xy$. If $xy = 2k\pi$, then $\cos xy = 1$ and $x = -2$, so $y = -k\pi$. If $xy = \pi + 2k\pi$, then $\cos xy = -1$ and $x = 2$, so $y = \frac{\pi}{2} + k\pi$. The solutions are

$$(-2, k\pi) \quad \text{and} \quad \left(2, \frac{\pi}{2} + k\pi\right), \quad k \text{ is an integer}$$

8. Find the largest and the smallest values of the function

$$x \mapsto f(x) = |x^2 - 6x| + |x^2 - 6x + 8|$$

if its domain is the set of real numbers satisfying $3 - \sqrt{10} \leq x \leq 7$. Where does the function take its extreme values?

Solution: $x^2 - 6x = x(x-6)$ and $x^2 - 6x + 8 = (x-2)(x-4)$. So

$$f(x) = \begin{cases} 2(x-3)^2 - 10 & \text{if } 3 - \sqrt{10} \leq x \leq 0 \text{ or } 6 \leq x \leq 7 \\ 8 & \text{if } 0 \leq x \leq 2 \text{ or } 4 \leq x \leq 6 \\ 10 - 2(x-3)^2 & \text{if } 2 \leq x \leq 4 \end{cases}$$

On $[3 - \sqrt{10}, 0]$ the values go from 10 down to 8; on $[2, 4]$ they are between 8 and 10; on $[6, 7]$ they go from 8 up to 22. So the largest value is 22, at $x = 7$, and the smallest value is 8, taken on the intervals $0 \leq x \leq 2$ and $4 \leq x \leq 6$.