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In memory of Imre Hajnal, teacher. Imre Hajnal (1926–1996) earned his diploma as a secondary school teacher of mathematics and physics at Eötvös Loránd University. He taught at the Bethlen Gábor High School in Hódmezővásárhely, at the Szakérttségis Kollégium in Szeged, and at the Ságvári Endre High School in Szeged, and also worked as a mathematics supervisor. He taught successfully in classes with a special mathematics curriculum. His high school textbooks, the teacher's guides written for them, and his other mathematics books will hopefully be used for a long time.

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sum of two sides of a triangle is 14 units, and the angles on the third side are 30° and 70° . Find the lengths of the sides, the area of the triangle, and the radii of the circumscribed and the inscribed circles.

Solution: The third angle is 80° . Let x be the side opposite the 30° angle; then $14 - x$ is opposite the 70° angle, and by the Law of Sines

$$\frac{x}{14 - x} = \frac{\sin 30^\circ}{\sin 70^\circ}, \quad x = \frac{14 \sin 30^\circ}{\sin 30^\circ + \sin 70^\circ} \approx 4.86$$

so the sides are about 4.86, 9.14, and (again by the Law of Sines) $\frac{4.86 \sin 80^\circ}{\sin 30^\circ} \approx 9.58$ units. The area is $\frac{1}{2} \cdot 4.86 \cdot 9.14 \sin 80^\circ \approx 21.88$ square units, the radius of the circumscribed circle is $r = \frac{x}{2 \sin 30^\circ} = x \approx 4.86$ units, and the radius of the inscribed circle is $\varrho = \frac{T}{s} \approx \frac{21.88}{11.79} \approx 1.86$ units.

2. The first term of an arithmetic sequence is 30, and its difference is -3 . Find the term of the sequence that is one eighth of the sum of the terms before it.

Solution: The condition is $8a_n = S_{n-1}$:

$$8(30 - 3(n - 1)) = \frac{n - 1}{2} (60 - 3(n - 2)), \quad n^2 - 39n + 198 = 0, \quad n = 6 \text{ or } n = 33$$

So the term is $a_6 = 15$ (and $30 + 27 + 24 + 21 + 18 = 120 = 8 \cdot 15$) or $a_{33} = -66$ (and $S_{32} = -528 = 8 \cdot (-66)$).

3. One parallel side of the trapezoid $ABCD$ is $DC = 8$ units, the legs are $AD = 4\sqrt{3}$ and $BC = 2\sqrt{10}$ units, and $\angle BAD = 60^\circ$. Find the area of the trapezoid.

Solution: The height is $m = 4\sqrt{3} \sin 60^\circ = 6$. The line through C parallel to AD meets AB at E ; then $AE = DC = 8$, $CE = 4\sqrt{3}$, and $\angle CEB = 60^\circ$. With $EB = x$, by the Law of Cosines in triangle CEB :

$$40 = 48 + x^2 - 2 \cdot 4\sqrt{3} \cdot x \cos 60^\circ, \quad x^2 - 4\sqrt{3}x + 8 = 0, \quad x = 2\sqrt{3} \pm 2$$

Both work, so there are two such trapezoids, with $AB = 10 + 2\sqrt{3}$ or $AB = 6 + 2\sqrt{3}$, and areas

$$\frac{8 + 10 + 2\sqrt{3}}{2} \cdot 6 = 54 + 6\sqrt{3} \approx 64.39 \quad \text{or} \quad \frac{8 + 6 + 2\sqrt{3}}{2} \cdot 6 = 42 + 6\sqrt{3} \approx 52.39$$

4. Solve the following system of equations over the pairs of real numbers.

$$2\sqrt{\frac{5x}{x-y}} + \sqrt{\frac{x-y}{5x}} = 3, \quad (y+4x)(x-y+20) = 0$$

Solution: We need $\frac{5x}{x-y} > 0$. By the second equation, $y = -4x$ or $x - y = -20$.

If $y = -4x$: $x - y = 5x$, and the first equation is $2 + 1 = 3$, true for every $x \neq 0$. So every pair $(t, -4t)$ with $t \neq 0$ is a solution.

If $x - y = -20$: let $u = \sqrt{\frac{x-y}{5x}} = \sqrt{-\frac{4}{x}}$; the first equation is $\frac{2}{u} + u = 3$, so $u = 1$ or $u = 2$, that is, $x = -4$ or $x = -1$. This gives $(-4, 16)$ (which is also in the first family) and $(-1, 19)$.

So the solutions are the pairs $(t, -4t)$ ($t \neq 0$) and the pair $(-1, 19)$.

5. Prove that the following relation holds in every triangle (with the usual notation).

$$\frac{b-c}{a} = \frac{\sin \frac{\beta-\gamma}{2}}{\sin \frac{\beta+\gamma}{2}}$$

Solution: By the Law of Sines and the sum-to-product formula, and since $\sin \alpha = \sin(\beta + \gamma) = 2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta + \gamma}{2}$:

$$\frac{b-c}{a} = \frac{\sin \beta - \sin \gamma}{\sin \alpha} = \frac{2 \cos \frac{\beta + \gamma}{2} \sin \frac{\beta - \gamma}{2}}{2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta + \gamma}{2}} = \frac{\sin \frac{\beta - \gamma}{2}}{\sin \frac{\beta + \gamma}{2}}$$

(we used $\cos \frac{\beta + \gamma}{2} \neq 0$, since $\beta + \gamma < 180^\circ$).

6. A line e through the origin cuts off a chord of length 8 from the circle $(x - 4)^2 + (y - 8)^2 = 32$. Find the equation of the line e , and the coordinates of the endpoints of the chord.

Solution: The radius is $4\sqrt{2}$, so a chord of length 8 is at distance $\sqrt{32 - 16} = 4$ from the center $(4, 8)$. A line through the origin is $Ax + By = 0$, and

$$\frac{|4A + 8B|}{\sqrt{A^2 + B^2}} = 4, \quad (A + 2B)^2 = A^2 + B^2, \quad B(4A + 3B) = 0$$

If $B = 0$: the line is $x = 0$, and the endpoints are $(0, 4)$ and $(0, 12)$.

If $4A + 3B = 0$: the line is $3x - 4y = 0$, and the endpoints are $\left(\frac{16}{5}, \frac{12}{5}\right)$ and $\left(\frac{48}{5}, \frac{36}{5}\right)$.

7. For which values of the parameter a does the equation

$$\sin^4 x + \cos^4 x + \sin 2x + a = 0$$

have a solution? Solve the equation for $a = \frac{1}{2}$ and for $a = -\frac{3}{2}$.

Solution: $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2 2x$. With $s = \sin 2x$, the equation is

$$1 - \frac{s^2}{2} + s + a = 0, \quad (s - 1)^2 = 3 + 2a$$

It has a solution with $-1 \leq s \leq 1$ if and only if $0 \leq 3 + 2a \leq 4$, that is, $-\frac{3}{2} \leq a \leq \frac{1}{2}$ (then $s = 1 - \sqrt{3 + 2a}$).

For $a = -\frac{3}{2}$: $\sin 2x = 1$, $x = \frac{\pi}{4} + k\pi$. For $a = \frac{1}{2}$: $\sin 2x = -1$, $x = \frac{3\pi}{4} + k\pi$ (k is an integer).

8. The parallel lines e and f are $4\sqrt{2}$ units apart. We take the points A and B on the line e so that $AB = 3\sqrt{2}$, and a point C on the line f . Let P be a point of the segment AC ; the line BP meets the line f at D . How far from the line e is the point P , if the sum of the areas of the triangles ABP and PCD is as small as possible? What is this smallest area?

Solution: Let x be the distance of P from e ($0 < x < 4\sqrt{2}$); then P is $4\sqrt{2} - x$ away from f . The triangles ABP and CDP are similar ($AB \parallel CD$), so $\frac{DC}{AB} = \frac{4\sqrt{2} - x}{x}$, and the sum of the areas is

$$T(x) = \frac{1}{2} \cdot 3\sqrt{2} \cdot x + \frac{1}{2} \cdot 3\sqrt{2} \cdot \frac{(4\sqrt{2} - x)^2}{x} = \frac{3\sqrt{2}}{2} \left(2x + \frac{32}{x} \right) - 24$$

By the inequality between the arithmetic and geometric means, $2x + \frac{32}{x} \geq 2\sqrt{64} = 16$, with equality for $x = 4$. So P is 4 units from e , and the smallest area is $24\sqrt{2} - 24 = 24(\sqrt{2} - 1) \approx 9.94$ square units.