

author: László Számádó

translated from: KoMaL, April 1998, pages 202-203

Pdf file: KoMaL, April 1998, pages 202-203

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\sqrt{x^2 - 10x + 30} + x^2 = 10x - 18$$

Solution: Let $y = x^2 - 10x + 30 = (x - 5)^2 + 5 \geq 5$. The equation is $\sqrt{y} = 12 - y$, which requires $y \leq 12$; squaring, $y^2 - 25y + 144 = 0$, so $y = 9$ or $y = 16$, and only $y = 9$ is allowed. Then $x^2 - 10x + 21 = 0$, so $x = 3$ or $x = 7$.

2. Seen from above, we see a regular hexagon $ABCDEF$ with side 10, but in reality the vertices are at heights 4, 6, 3, 6, 2, 3 units above the horizontal ground, in this order. Seen from above, the lines AD , BE , and CF seem to meet at one point. In reality, what are the vertical distances between these lines at the apparent intersection point?

Solution: Seen from above, the three lines meet at the center O of the hexagon, which is the midpoint of each of the segments AD , BE , CF . Along each segment the height changes linearly, so above O the height of each line is the average of the heights of its endpoints:

$$AD : \frac{4+6}{2} = 5, \quad BE : \frac{6+2}{2} = 4, \quad CF : \frac{3+3}{2} = 3$$

So at the apparent intersection point the line AD is highest and CF is lowest: the vertical distance is 1 unit between AD and BE , 1 unit between BE and CF , and 2 units between AD and CF .

3. The four-digit number $\overline{abbc}$ (in base ten) is a prime. Different letters denote different digits. The sum of the four digits is 26. If we cut the number into two parts with one cut, then among the numbers obtained, $\overline{abb}$, $\overline{c}$, $\overline{ab}$, $\overline{bc}$, and also $\overline{bbc}$ are primes. Find all four-digit numbers with these properties.

Solution: c is a prime digit: 2, 3, 5, or 7; and $\overline{bc}$ is a prime, so c is not 2 or 5: $c = 3$ or $c = 7$. $\overline{ab}$ is a prime, so b is 1, 3, 7, or 9. Since $a + 2b + c = 26$ with $a \leq 9$ and $c \leq 7$, we need $2b \geq 10$, so $b = 7$ or $b = 9$; then $\overline{bbc}$ is 773, 777, 993, or 997, and 777 = 3 · 7 · 37, 993 = 3 · 331 are not primes. So $(b, c) = (7, 3)$ with $a = 9$, or $(b, c) = (9, 7)$ with $a = 1$: the candidates are 9773 and 1997. But 9773 = 29 · 337, while 1997 is a prime, and so are 199, 7, 19, 97, 997. The only such number is 1997.

4. In the coordinate plane, the points $A(0, 0)$, $B(6, 0)$, $D(0, 6)$, $P(2, 0)$, $S(0, 4)$, $E(9, 0)$, $F(0, 9)$ are given. The point C is an interior point of the segment EF . The point of the segment CD dividing it into thirds that is closer to C is R , and the point of the segment BC dividing it into thirds that is closer to B is Q . The midpoints of the segments AC , BD , PR , QS are K , L , M , N . Find the ratio $KL : MN$ as a function of the coordinates of the point C .

Solution: $C = (c, 9 - c)$ with $0 < c < 9$. Then $R = \frac{2C + D}{3} = \left(\frac{2c}{3}, \frac{24 - 2c}{3}\right)$ and $Q = \frac{2B + C}{3} = \left(\frac{12 + c}{3}, \frac{9 - c}{3}\right)$, so

$$K\left(\frac{c}{2}, \frac{9 - c}{2}\right), L(3, 3), M\left(\frac{c + 3}{3}, \frac{12 - c}{3}\right), N\left(\frac{12 + c}{6}, \frac{21 - c}{6}\right)$$

and $\overrightarrow{LK} = \left(\frac{c - 6}{2}, \frac{3 - c}{2}\right)$, $\overrightarrow{NM} = \left(\frac{c - 6}{6}, \frac{3 - c}{6}\right)$. So $\overrightarrow{LK} = 3\overrightarrow{NM}$, and $KL : MN = 3$, which does not depend on C .

5. The function f defined on the real numbers is given by $f(a) = 2a^3 - 3a^2 + a$. Solve the following equation over the real numbers.

$$f(\sin^2 x) + f(\cos^2 x) = 0$$

Solution: $f(a) = a(2a - 1)(a - 1)$, and so $f(1 - a) = (1 - a)(1 - 2a)(-a) = -f(a)$. Since $\cos^2 x = 1 - \sin^2 x$, the left-hand side is $f(\sin^2 x) - f(\sin^2 x) = 0$ for every x . So every real number is a solution.

6. Solve the following equation over the real numbers.

$$\left(\frac{\sqrt{5}}{7}\right)^{\frac{2}{-5 + \log x} - \frac{4}{1 + \log x}} = 9.8$$

Solution: We need $x > 0$, $x \neq 10^5$, $x \neq 10^{-1}$. Since $9.8 = \frac{49}{5} = \left(\frac{\sqrt{5}}{7}\right)^{-2}$, the equation means

$$\frac{2}{\log x - 5} - \frac{4}{\log x + 1} = -2$$

With $L = \log x$: $2(L + 1) - 4(L - 5) = -2(L - 5)(L + 1)$, that is, $L^2 - 5L + 6 = 0$, so $L = 2$ or $L = 3$. The solutions are $x = 100$ and $x = 1000$.

7. Two circles that are not in the same plane intersect each other in two points. Prove that there is always a sphere that contains both circles.

Solution: Let the two common points be A and B , and choose a point $C \neq A, B$ on the first circle and a point $D \neq A, B$ on the second circle. The circles are not in one plane, so A, B, C, D are not in one plane, and they form a tetrahedron. Every tetrahedron has a circumscribed sphere. The plane of the first circle cuts this sphere in a circle through A, B, C ; but there is only one circle through three points, so this is the first circle. Similarly, the plane of the second circle cuts the sphere in the second circle. So the sphere contains both circles.

8. Three gold diggers found a nugget every day, which weighed as many grams as the number of the day they were working (on the first day 1 gram, on the second day 2 grams, and so on). On which day can they first share the gold equally, if they do not want to cut the nuggets?

Solution: After n days they have $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ grams, which must be divisible by 3; so $3 \mid n$ or $3 \mid n+1$. For $n = 2$ and $n = 3$ this is not enough ($3 = 1 + 2$ can not be split into three equal parts, and neither can 6 with the nuggets 1, 2, 3). For $n = 5$ it works: $1 + 4 = 2 + 3 = 5$. So they can first share the gold equally on the fifth day.

(In fact they can share it exactly on the days $n \geq 5$ with $3 \mid n$ or $3 \mid n+1$: for $n = 5, 6, 8, 9$ there are direct splittings, for example $1 + 6 = 2 + 5 = 3 + 4$ and $1 + 2 + 3 + 6 = 4 + 8 = 5 + 7$ and $1 + 2 + 3 + 4 + 5 = 6 + 9 = 7 + 8$; and any six further consecutive nuggets $k + 1, \dots, k + 6$ can be split into the three pairs $(k + 1) + (k + 6) = (k + 2) + (k + 5) = (k + 3) + (k + 4)$.)