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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$5^x \cdot 2^{\frac{x}{x-1}} = 100$$

Solution: We need $x \neq 1$. Dividing by $100 = 5^2 \cdot 2^2$, and using $\frac{x}{x-1} - 2 = \frac{2-x}{x-1} = \frac{x-2}{1-x}$:

$$5^{x-2} \cdot 2^{\frac{x-2}{1-x}} = 1, \quad \left(5 \cdot 2^{\frac{1}{1-x}}\right)^{x-2} = 1$$

So $x = 2$, or $5 \cdot 2^{\frac{1}{1-x}} = 1$, that is, $\frac{1}{1-x} = -\log_2 5$, $x = 1 + \log_5 2 = \log_5 10 \approx 1.43$. The solutions are $x = 2$ and $x = \log_5 10$.

2. Prove that if in a triangle

$$\frac{a-b}{a} = 1 - 2 \cos \gamma$$

then the triangle is isosceles.

Solution: The condition says $\frac{b}{a} = 2 \cos \gamma$, and by the Law of Cosines $2 \cos \gamma = \frac{a^2 + b^2 - c^2}{ab}$. So

$$\frac{b}{a} = \frac{a^2 + b^2 - c^2}{ab}, \quad b^2 = a^2 + b^2 - c^2, \quad c = a$$

so the triangle is isosceles.

3. Solve the following inequality over the real numbers.

$$\frac{x^2}{(2 - \sqrt{x+4})^2} > x + 6$$

Solution: We need $x \geq -4$ and $\sqrt{x+4} \neq 2$, that is, $x \neq 0$. Since $x = (\sqrt{x+4} - 2)(\sqrt{x+4} + 2)$, the left-hand side is $(2 + \sqrt{x+4})^2 = x + 8 + 4\sqrt{x+4}$, and the inequality is $2 + 4\sqrt{x+4} > 0$, which is always true. So the solution is every $x \geq -4$ with $x \neq 0$.

4. For which values of the variables do the following expressions take their largest values? Find these largest values.

$$\text{a) } \frac{4\sqrt{2} + 2}{2^{x-1} + 2^{2-x} + 1} \qquad \text{b) } \frac{32}{(3 - x + \log 2y)^2 + x^4 - 2x^2 + 17}$$

Solution: In both cases the numerator is a positive constant and the denominator is positive, so the fraction is largest where the denominator is smallest.

a) By the inequality between the arithmetic and geometric means, $2^{x-1} + 2^{2-x} \geq 2\sqrt{2^{x-1} \cdot 2^{2-x}} = 2\sqrt{2}$, with equality for $2^{x-1} = 2^{2-x}$, that is, $x = \frac{3}{2}$. The largest value is $\frac{4\sqrt{2} + 2}{2\sqrt{2} + 1} = 2$, at $x = \frac{3}{2}$.

b) The denominator is $(3 - x + \log 2y)^2 + (x^2 - 1)^2 + 16 \geq 16$, with equality if $x^2 = 1$ and $\log 2y = x - 3$. The largest value is $\frac{32}{16} = 2$, at $(x, y) = (1, 0.005)$ and $(x, y) = (-1, 0.00005)$.

5. The difference of an arithmetic sequence is $\frac{1}{2}$. The sum of the first n terms is $\frac{75}{2}$, and the sum of the first $n + 3$ terms is 63. Find n and the first term of the sequence.

Solution: The terms $a_{n+1} + a_{n+2} + a_{n+3} = 63 - \frac{75}{2} = \frac{51}{2}$, that is, $3a_1 + (3n + 3) \cdot \frac{1}{2} = \frac{51}{2}$, so $2a_1 + n = 16$. Also $\frac{n}{2} \left(2a_1 + \frac{n-1}{2} \right) = \frac{75}{2}$, that is, $n(4a_1 + n - 1) = 150$. With $2a_1 = 16 - n$:

$$n(31 - n) = 150, \quad n^2 - 31n + 150 = 0, \quad n = 6 \text{ or } n = 25$$

If $n = 6$, then $a_1 = 5$; if $n = 25$, then $a_1 = -\frac{9}{2}$. Both work.

6. Solve the following system of equations over the triples of integers.

$$x^2 - y^2 - z^2 = 1, \quad y + z - x = 3$$

Solution: From the second equation $x = y + z - 3$. Substituting into the first one: $(y + z - 3)^2 - y^2 - z^2 = 1$, which simplifies to $yz - 3y - 3z = -4$, that is,

$$(y - 3)(z - 3) = 5$$

So $(y - 3, z - 3)$ is $(1, 5)$, $(5, 1)$, $(-1, -5)$, or $(-5, -1)$, and the solutions (x, y, z) are

$$(9, 4, 8), \quad (9, 8, 4), \quad (-3, 2, -2), \quad (-3, -2, 2)$$

7. The circle k is tangent to the x -axis, and also to the circle $(x - 17)^2 + (y + 17)^2 = 100$ at its point with first coordinate $x_0 = 11$. Find the equation of the circle k .

Solution: The given circle has center $C(17, -17)$ and radius 10; its points with first coordinate 11 are $E_1(11, -9)$ and $E_2(11, -25)$. The center of k is on the line through C and the point of tangency, and k touches the common tangent at that point.

For E_1 : the tangent to the given circle at E_1 (normal vector $\overrightarrow{E_1C}$, parallel to $(3, -4)$) is $3x - 4y = 69$, which meets the x -axis at $M(23, 0)$. The tangent segments from M are equal: $ME_1 = 15$, so k touches the x -axis at $(8, 0)$ or $(38, 0)$. The center is on the line CE_1 : $4x + 3y = 17$, which gives the circles

$$(x - 8)^2 + (y + 5)^2 = 5^2 \quad \text{and} \quad (x - 38)^2 + (y + 45)^2 = 45^2$$

For E_2 , in the same way (the tangent is $3x + 4y = -67$, which meets the x -axis at $(-\frac{67}{3}, 0)$, $\frac{125}{3}$ away from E_2):

$$(x + 64)^2 + (y + 125)^2 = 125^2 \quad \text{and} \quad \left(x - \frac{58}{3}\right)^2 + \left(y + \frac{125}{9}\right)^2 = \left(\frac{125}{9}\right)^2$$

So there are four such circles.

8. Depending on the value of the real parameter m , how many different real roots does the following equation have?

$$x^4 + 2(m^2 - 5)x^2 + m^4 - 10m^2 + 7 = 0$$

Solution: This is a quadratic equation in x^2 , with $\frac{D}{4} = (m^2 - 5)^2 - (m^4 - 10m^2 + 7) = 18$, so

$$x^2 = 5 + 3\sqrt{2} - m^2 \quad \text{or} \quad x^2 = 5 - 3\sqrt{2} - m^2$$

Each of these gives two roots if the right-hand side is positive, one root ($x = 0$) if it is 0, and none if it is negative. With $A = \sqrt{5 - 3\sqrt{2}} \approx 0.87$ and $B = \sqrt{5 + 3\sqrt{2}} \approx 3.04$, the number of different real roots is

- 4 if $|m| < A$,
- 3 if $|m| = A$,
- 2 if $A < |m| < B$,
- 1 if $|m| = B$,
- 0 if $|m| > B$.

Note: In KoMaL, “= 0” is missing from the end of the equation.