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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\frac{x-2}{x-1} + \frac{x+2}{x+1} = \frac{x-4}{x-3} + \frac{x+4}{x+3} - \frac{28}{15}$$

Solution: We need $x \neq \pm 1, \pm 3$. Adding the fractions on each side:

$$\frac{2x^2-4}{x^2-1} = \frac{2x^2-24}{x^2-9} - \frac{28}{15}$$

With $u = x^2$ and multiplying by $15(u-1)(u-9)$, this simplifies to $7u^2 - 55u + 108 = 0$, so $u = 4$ or $u = \frac{27}{7}$. The solutions are

$$x = \pm 2 \quad \text{and} \quad x = \pm 3\sqrt{\frac{3}{7}} \approx \pm 1.96$$

2. Prove that in every triangle

$$\sin^2 \alpha = \sin^2 \beta + \sin^2 \gamma - 2 \sin \beta \sin \gamma \cos \alpha$$

where α, β, γ are the angles of the triangle.

Solution: By the Law of Sines $\sin \alpha = \frac{a}{2r}$, $\sin \beta = \frac{b}{2r}$, $\sin \gamma = \frac{c}{2r}$, where r is the radius of the circumscribed circle. So the right-hand side is

$$\frac{b^2 + c^2 - 2bc \cos \alpha}{4r^2} = \frac{a^2}{4r^2} = \sin^2 \alpha$$

by the Law of Cosines.

3. Find the values of the parameter a for which the inequality

$$-2 < \frac{x^2 + ax - 2}{-x^2 + x - 1} < 3$$

holds for every real number x .

Solution: The denominator $-x^2 + x - 1 = -\left(x - \frac{1}{2}\right)^2 - \frac{3}{4}$ is always negative, so multiplying by it reverses the inequalities:

$$-3x^2 + 3x - 3 < x^2 + ax - 2 < 2x^2 - 2x + 2$$

that is, $4x^2 + (a - 3)x + 1 > 0$ and $x^2 - (a + 2)x + 4 > 0$ for every x . This holds if and only if both discriminants are negative:

$$(a - 3)^2 < 16 \quad \text{and} \quad (a + 2)^2 < 16, \quad -1 < a < 7 \quad \text{and} \quad -6 < a < 2$$

So the inequality holds for every x exactly when $-1 < a < 2$.

4. The base of the isosceles triangle ABC is $BC = 4.8$ units, and the altitude to it is $m = AD = 4.5$ units. The point P is the point of the leg AC for which the quadrilateral $ABDP$ is a cyclic quadrilateral. Find the length of the segment AP .

Solution: D is the midpoint of BC , so $CD = 2.4$, and $CA = \sqrt{2.4^2 + 4.5^2} = 5.1$. The lines CA and CB are secants of the circle through A, B, D, P , so by the power of the point C :

$$CP \cdot CA = CD \cdot CB, \quad CP = \frac{2.4 \cdot 4.8}{5.1} \approx 2.26$$

and $AP = 5.1 - CP \approx 2.84$ units.

5. The sums of the first n, k , and $n + k$ terms of an arithmetic sequence are S_n, S_k , and S_{n+k} . Prove that

$$(n + k)(S_n - S_k) = (n - k)S_{n+k}$$

Solution: With the first term a and difference d , $S_m = \frac{m}{2}(2a + (m - 1)d) = ma + \frac{m(m - 1)}{2}d$. So

$$S_n - S_k = (n - k)a + \frac{n^2 - n - k^2 + k}{2}d = \frac{n - k}{2}(2a + (n + k - 1)d)$$

and $(n + k)(S_n - S_k) = (n - k) \cdot \frac{n + k}{2}(2a + (n + k - 1)d) = (n - k)S_{n+k}$.

6. Two vertices of triangle ABC are $A(0, 5)$ and $B(-4, 0)$, and the area of the triangle is 9 square units. Find the first coordinate of the vertex C , if its second coordinate is 7.

Solution: $AB = \sqrt{41}$, so the distance of C from the line AB is $\frac{18}{\sqrt{41}}$. The line AB is $5x - 4y + 20 = 0$, so for $C(x, 7)$:

$$\frac{|5x - 28 + 20|}{\sqrt{41}} = \frac{18}{\sqrt{41}}, \quad 5x - 8 = \pm 18$$

So the first coordinate of C is $\frac{26}{5}$ or -2 .

Note: In KoMaL, the vertex is called X in this problem; KoMaL corrected this to C in the next issue.

7. Find the values of the parameter a for which one root of the equation

$$2x^2 - 2(2a + 1)x + a(a - 1) = 0$$

is less than a , and the other root is greater than a .

Solution: The leading coefficient is positive, so this happens exactly when the quadratic expression is negative at $x = a$:

$$2a^2 - 2(2a + 1)a + a(a - 1) = -a^2 - 3a < 0, \quad a(a + 3) > 0$$

So $a < -3$ or $a > 0$.

8. For which values of the parameter α does the equation

$$x^2 - 2(\sin \alpha + \cos \alpha)x + 1 + 2\cos^2 \alpha = 0$$

have a) two equal, b) two different real roots?

Solution: $\frac{D}{4} = (\sin \alpha + \cos \alpha)^2 - 1 - 2\cos^2 \alpha = 2\sin \alpha \cos \alpha - 2\cos^2 \alpha = 2\cos \alpha (\sin \alpha - \cos \alpha)$.

a) $D = 0$ if $\cos \alpha = 0$ or $\tan \alpha = 1$: $\alpha = \frac{\pi}{2} + k\pi$ or $\alpha = \frac{\pi}{4} + k\pi$ (then the equation is $(x \mp 1)^2 = 0$ or $(x \mp \sqrt{2})^2 = 0$).

b) $D > 0$ if $\cos \alpha (\sin \alpha - \cos \alpha) > 0$, that is, $\cos^2 \alpha (\tan \alpha - 1) > 0$, $\tan \alpha > 1$:

$$\frac{\pi}{4} + k\pi < \alpha < \frac{\pi}{2} + k\pi, \quad k \text{ is an integer}$$