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translated from: KoMaL, April 1999, page 209

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. For which value of n are a, b, c_n three consecutive terms of an arithmetic sequence, if

$$a = \tan \frac{0 \cdot \pi}{5} + \tan \frac{1 \cdot \pi}{5} + \tan \frac{2 \cdot \pi}{5} + \tan \frac{3 \cdot \pi}{5} + \tan \frac{4 \cdot \pi}{5} + \tan \frac{5 \cdot \pi}{5}, \quad b = \sin^{\cos 60^\circ} 60^\circ + \cos^{\sin 30^\circ} 30^\circ$$

and $c_1 = 3, c_n = 2\sqrt{c_{n-1}}$ for $n \geq 2$?

Solution: Since $\tan(\pi - \varphi) = -\tan \varphi$, the terms of a cancel in pairs (the first with the last, the second with the fifth, the third with the fourth), so $a = 0$. Also $\cos 60^\circ = \sin 30^\circ = \frac{1}{2}$ and $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, so $b = 2\sqrt{\frac{\sqrt{3}}{2}} = \sqrt{2\sqrt{3}}$.

The numbers $0, b, c_n$ form an arithmetic sequence if $c_n = 2b = 2\sqrt{2\sqrt{3}}$. Now $c_1 = 3, c_2 = 2\sqrt{3}, c_3 = 2\sqrt{2\sqrt{3}}$, and the sequence (c_n) is strictly increasing (it increases towards 4), so $n = 3$.

2. In the quadratic equation $5x^2 - 6(a+b)x + a^2 + b^2 + 2ab = 0$ the real parameters a and b satisfy $1 < a + b < 5$. Show that one root of the equation is greater than 0 but less than 1, and the other root is greater than 1. Is the converse of this statement true?

Solution: Let $s = a + b$; the equation is $5x^2 - 6sx + s^2 = 0$, that is, $(x - s)(5x - s) = 0$, with roots s and $\frac{s}{5}$. If $1 < s < 5$, then $s > 1$ and $0 < \frac{s}{5} < 1$.

The converse is also true: if one root is between 0 and 1 and the other is greater than 1, then both roots are positive, so $s > 0$ and $\frac{s}{5} < s$; so the smaller root is $\frac{s}{5} < 1$ and the larger is $s > 1$, which gives $1 < s < 5$.

3. Find the integers n and m for which

$$A = n^5 - 5n^3 + 4n - 2^m + 1$$

is divisible by 5.

Solution: $n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)n(n + 1)(n + 2)$ is the product of five consecutive integers, so it is divisible by 5 for every n . So we need $2^m - 1$ to be divisible by 5 (and A to be an integer, so $m \geq 0$). The last digits of the powers $2^1, 2^2, \dots$ repeat as 2, 4, 8, 6, so $2^m \equiv 1$ modulo 5 exactly when m is divisible by 4. The solutions are: n is any integer, and m is any nonnegative multiple of 4.

4. The center of the inscribed circle of a right triangle is at distances x, y, z from the vertices (z is the distance from the vertex of the right angle). Show that one angle of the triangle with sides xy, xz , and yz is 135° .

Solution: Let the triangle be ABC with the right angle at C , the center of the inscribed circle O and its radius r , with $OA = x, OB = y, OC = z$. OC bisects the right angle, so $z = \sqrt{2}r$. In the triangle AOB , the angle at O is $180^\circ - \frac{\alpha + \beta}{2} = 135^\circ$, and its area is

$$\frac{1}{2}xy \sin 135^\circ = \frac{1}{2} \cdot AB \cdot r, \quad AB = \frac{xy}{\sqrt{2}r} = \frac{xy}{z}$$

By the Law of Cosines in triangle AOB , $AB^2 = x^2 + y^2 - 2xy \cos 135^\circ$. Multiplying by z^2 :

$$(xy)^2 = (xz)^2 + (yz)^2 - 2(xz)(yz) \cos 135^\circ$$

So by the Law of Cosines, in the triangle with sides xy, xz, yz the angle opposite the side xy is 135° .

5. Find the equation of the circle that the circles $(x+2)^2 + (y-5)^2 = 1$, $(x-1)^2 + (y-6)^2 = 1$, and $(x-4)^2 + (y+3)^2 = 1$ touch from the inside.

Solution: The centers are $A(-2, 5)$, $B(4, -3)$, $C(1, 6)$, and all three radii are 1. The center of the circle we are looking for is equally far from A, B, C , so it is the center of the circle circumscribed around the triangle ABC , and its radius is 1 more than the radius of that circle. Since $AB = 10$, $BC = 3\sqrt{10}$, $CA = \sqrt{10}$, and $AB^2 = BC^2 + CA^2$, the triangle has a right angle at C , so the circumscribed circle has center $F(1, 1)$ (the midpoint of AB) and radius 5. The circle is

$$(x-1)^2 + (y-1)^2 = 36$$

6. The numbers a, b , and c are (not necessarily consecutive) terms of an arithmetic sequence of positive numbers, and

$$\frac{c-b}{a} + \frac{a-c}{b} + \frac{b-a}{c} = 0$$

Find the sum of the first n terms of the sequence, if we know its first term.

Solution: Multiplying by abc , and factoring the left-hand side (this can be checked by expanding):

$$bc^2 - b^2c + a^2c - ac^2 + ab^2 - a^2b = (c-b)(a-c)(a-b) = 0$$

So two of the numbers a, b, c are equal. Two different terms of an arithmetic sequence can be equal only if the difference is 0, so all terms are equal to the first term a_1 , and $S_n = na_1$.

(We assume that a, b, c are at different positions in the sequence.)

7. Consider the functions $f(x) = \sin^4 x + \cos^4 x$ and $g(x) = \cos x$. Give a geometric transformation that takes the graph of g to the graph of f .

$$\text{Solution: } \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{\sin^2 2x}{2} = 1 - \frac{1 - \cos 4x}{4} = \frac{3}{4} + \frac{\cos 4x}{4}.$$

The dilation with center at the origin and ratio $\frac{1}{4}$ takes the point $(t, \cos t)$ to $\left(\frac{t}{4}, \frac{\cos t}{4}\right)$, that is, the graph of $\cos x$ to the graph of $\frac{\cos 4x}{4}$. Then the translation by the vector $\left(0, \frac{3}{4}\right)$ takes it to the graph of f . So the composition of these two transformations (which is itself a dilation, with center $(0, 1)$ and ratio $\frac{1}{4}$) takes the graph of g to the graph of f .

8. Solve the following system of equations over the real numbers.

$$\sqrt[3]{x-72} + \sqrt[3]{y+1998} = 25, \quad x + y = 1999$$

Solution: Let $a = \sqrt[3]{x-72}$ and $b = \sqrt[3]{y+1998}$. Then $a + b = 25$ and $a^3 + b^3 = (x-72) + (y+1998) = 3925$. Since $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$, $15625 - 75ab = 3925$, so $ab = 156$, and a, b are 12 and 13. The solutions are

$$(x, y) = (1800, 199) \quad \text{and} \quad (x, y) = (2269, -270)$$