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In memory of Gyula Komlós, teacher (my friend)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The medians of a triangle are 2.4, 2.6, and 1 units long. Find the area of the triangle.

Solution: Let S be the centroid of triangle ABC , C_1 the midpoint of AB , and S_c the reflection of S in C_1 . Then $ASBS_c$ is a parallelogram, and the sides of the triangle ASS_c are AS , $AS_c = BS$, and $SS_c = 2SC_1 = CS$: two thirds of the three medians, that is, 1.6 , $\frac{2}{3} \cdot 2.6$, and $\frac{2}{3}$. Its area is equal to the area of the triangle ASB (both are half of the parallelogram $ASBS_c$), which is one third of the area T of triangle ABC . Since $1^2 + 2.4^2 = 2.6^2$, the triangle ASS_c has a right angle, and

$$\frac{T}{3} = \frac{1}{2} \cdot \frac{2}{3} \cdot 1 \cdot \frac{2}{3} \cdot 2.4, \quad T = 1.6 \text{ square units}$$

2. Find the values of the real parameter a for which one root of the equation

$$x^2 - 4x + (3 + 2a - a^2) = 0$$

is twice the other root.

Solution: The discriminant is $16 - 4(3 + 2a - a^2) = 4(a - 1)^2$, so the roots are $x_1 = a + 1$ and $x_2 = 3 - a$. If $a + 1 = 2(3 - a)$, then $a = \frac{5}{3}$; if $3 - a = 2(a + 1)$, then $a = \frac{1}{3}$. In both cases the roots are $\frac{4}{3}$ and $\frac{8}{3}$. So $a = \frac{5}{3}$ or $a = \frac{1}{3}$.

3. Solve the following system of equations over the pairs of real numbers.

$$x^{\log_2 y} + y^{\log_2 x} = 2, \quad \log_4 x - \log_4 y = \frac{1}{2}$$

Solution: We need $x > 0$, $y > 0$. Since $x^{\log_2 y} = 2^{\log_2 x \log_2 y} = y^{\log_2 x}$, the first equation says $x^{\log_2 y} = 1$, that is, $\log_2 x \cdot \log_2 y = 0$. The second equation says $\frac{x}{y} = 2$. If $\log_2 x = 0$, then $x = 1$, $y = \frac{1}{2}$; if $\log_2 y = 0$, then $y = 1$, $x = 2$. The solutions are $\left(1, \frac{1}{2}\right)$ and $(2, 1)$.

4. For which values of the real parameter m is the value of the expression

$$(5 - m)x^2 - 3x + m + 3$$

positive for every real number x ?

Solution: If $m = 5$, the expression is $-3x + 8$, which also takes negative values. Otherwise it is positive for every x if and only if $5 - m > 0$ and the discriminant is negative:

$$9 - 4(5 - m)(m + 3) < 0, \quad 4m^2 - 8m - 51 < 0, \quad (m - 1)^2 < \frac{55}{4}$$

So $1 - \frac{\sqrt{55}}{2} < m < 1 + \frac{\sqrt{55}}{2}$, and since $1 + \frac{\sqrt{55}}{2} \approx 4.71 < 5$, this is the answer.

5. Consider the triangles whose area is $T = 4\sqrt{3}$ square units and one of whose angles is 60° .

a) In which of these triangles is the sum of the sides enclosing the 60° angle the smallest? What are the sides and angles of this triangle?

b) In which of these triangles is the side opposite the 60° angle the shortest?

Solution: Let the sides enclosing the 60° angle be a and b , and the third side c . Then $\frac{ab \sin 60^\circ}{2} = 4\sqrt{3}$, so $ab = 16$.

a) By the inequality between the arithmetic and geometric means, $a + b \geq 2\sqrt{ab} = 8$, with equality for $a = b = 4$. Then the triangle is equilateral: all sides are 4 and all angles are 60° .

b) By the Law of Cosines $c^2 = a^2 + b^2 - ab = a^2 + b^2 - 16 \geq 2ab - 16 = 16$, with equality for $a = b$. So c is shortest ($c = 4$) again for the equilateral triangle.

6. Let $A(1, 0)$, $B(0, 4)$, and $C(c, 6)$. Find the value of c , if the area of the triangle is 13 square units.

Solution: The area is $\frac{1}{2}|(-1) \cdot 6 - 4(c - 1)| = \frac{1}{2}|4c + 2|$ (using $\overrightarrow{AB} = (-1, 4)$ and $\overrightarrow{AC} = (c - 1, 6)$). So $|4c + 2| = 26$, and $c = 6$ or $c = -7$.

7. Solve the following inequality over the real numbers.

$$\sin^4 x + \cos^4 x \geq \frac{5}{8}$$

Solution: $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x = 1 - \frac{\sin^2 2x}{2} = 1 - \frac{1 - \cos 4x}{4} = \frac{3}{4} + \frac{\cos 4x}{4}$. So the inequality is $\cos 4x \geq -\frac{1}{2}$, that is, $-\frac{2\pi}{3} + 2k\pi \leq 4x \leq \frac{2\pi}{3} + 2k\pi$:

$$-\frac{\pi}{6} + \frac{k\pi}{2} \leq x \leq \frac{\pi}{6} + \frac{k\pi}{2}, \quad k \text{ is an integer}$$

8. Solve the equation $\sqrt{a-x} + \sqrt{-x} = 1$ over the real numbers, where a is a real parameter.

Solution: Suppose x is a solution. Then $\sqrt{a-x} = 1 - \sqrt{-x}$, and squaring, $a-x = 1 - x - 2\sqrt{-x}$, so $\sqrt{-x} = \frac{1-a}{2}$, and $x = -\left(\frac{1-a}{2}\right)^2$ (this requires $a \leq 1$). Substituting back:

$$\sqrt{a + \left(\frac{a-1}{2}\right)^2} + \sqrt{\left(\frac{a-1}{2}\right)^2} = \frac{|a+1|}{2} + \frac{|a-1|}{2} = 1$$

which holds exactly when $-1 \leq a \leq 1$. So for $-1 \leq a \leq 1$ the only solution is $x = -\left(\frac{a-1}{2}\right)^2$, and for other values of a there is no solution.