Problem Set III - 1999 (February 1999)

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In honor of the Janus Pannonius High School in Pécs, where I taught in the 1956–57 school year.
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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

   a) $2^x+\sqrt{2^{x+2}\cdot 5^x}=3\cdot 5^x$ b) $\dfrac{1}{x^2-3x+3}+\dfrac{2}{x^2-3x+4}=\dfrac{6}{x^2-3x+5}$ c) $2\sin 2x+2\sin x=2\cos x+1$

2. Solve the following inequalities over the real numbers.

   a) $(x+1)(|x|-1)\le -\dfrac{1}{2}$ b) $\tan x+\dfrac{1}{\tan x}\ge -2$ c) $\log_2(4^x-5\cdot 2^x+8)>2$

3. Consider the line e with equation $4x-3y+20=0$ and its point E with first coordinate $x_0=-2$. The circle k has radius 10 and is tangent to the line e at the point E.

   a) Find the coordinates of the vectors $\overrightarrow{EK}$ and $\overrightarrow{OK}$, where K is the center of the circle k and O is the origin.

   b) Write down the equation of the circle k.

4. Prove that the ratio of the two roots of the equation $ax^2+bx+c=0$ ($a\neq 0$, $b^2-4ac>0$) is $3:2$ if and only if $6b^2=25ac$.

5. In the rectangle $ABCD$, $2\cdot AB=3\cdot BC$. The distances of the vertices C and D from a point P of the line AB are $8\sqrt{2}$ and $4\sqrt{5}$ units. Find the sides of the rectangle. Where is the point P?

6. Prove that a triangle is a right triangle if and only if $\sin\gamma=\cos\alpha+\cos\beta$, where γ is an acute angle of the triangle.

7. Solve the inequality $x^{1+\log_a x}>a^2x$, where $a>0$, $a\neq 1$ is a real parameter.

8. Find the smallest value of the expression

$$f(x,y)=x^4+4y^4+\frac{1}{x^2y^2}$$

   For which pairs (x,y) does $f(x,y)$ take its smallest value?

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