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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. One parallel side of a trapezoid is 22 units long, its legs are 14.3 and 16.5 units long, and its height is 13.2 units. Find the area of the trapezoid.

Solution: The projections of the legs onto the line of the parallel sides are $\sqrt{14.3^2 - 13.2^2} = 5.5$ and $\sqrt{16.5^2 - 13.2^2} = 9.9$. Each leg can lean inward or outward, so the other parallel side can be

$$22 - 5.5 - 9.9 = 6.6, \quad 22 + 5.5 - 9.9 = 17.6, \quad 22 - 5.5 + 9.9 = 26.4, \quad 22 + 5.5 + 9.9 = 37.4$$

So there are four such trapezoids, and their areas $\frac{22 + c}{2} \cdot 13.2$ are

$$188.76, \quad 261.36, \quad 319.44, \quad 392.04 \text{ square units}$$

(The solution sketch in KoMaL gives 313.44 for the third area; the correct value is $\frac{22 + 26.4}{2} \cdot 13.2 = 319.44$.)

2. Prove that if n is a positive integer, then the following expressions are divisible by 24.

a) $(n^2 + 3n + 1)^2 - 1$ b) $n^4 + 6n^3 + 11n^2 + 6n$

Solution: Both expressions are equal to $n(n+1)(n+2)(n+3)$: in a)

$$(n^2 + 3n + 1)^2 - 1 = (n^2 + 3n)(n^2 + 3n + 2) = n(n+3)(n+1)(n+2)$$

and b) can be checked by expanding. Among four consecutive integers there are two consecutive even numbers, one of which is divisible by 4, so the product is divisible by 8; and one of them is divisible by 3. So the product is divisible by 24.

3. Find the places of the extreme values and the range of the function $x \mapsto \frac{1}{\cos^2 x + \sin^4 x}$ (x real).

Solution: Using $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$:

$$\cos^2 x + \sin^4 x = \frac{1 + \cos 2x}{2} + \frac{(1 - \cos 2x)^2}{4} = \frac{3}{4} + \frac{1}{4} \cos^2 2x$$

which takes every value from $\frac{3}{4}$ to 1. So the range of the function is $\left[1, \frac{4}{3}\right]$. The smallest value 1 is taken where

$\cos^2 2x = 1$, that is, at $x = \frac{k\pi}{2}$, and the largest value $\frac{4}{3}$ where $\cos 2x = 0$, that is, at $x = \frac{\pi}{4} + \frac{k\pi}{2}$ (k is an integer).

4. The numbers a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are the first four terms of two geometric sequences, with $a_1 = b_1$, $a_2 - b_2 = 3$, $a_3 - b_3 = 6$, and $a_4 - b_4 = 36$. Write down the first four terms of the sequences.

Solution: Let $a_k = aq^{k-1}$; then $b_1 = a$, $b_2 = aq - 3$, $b_3 = aq^2 - 6$, $b_4 = aq^3 - 36$. For a geometric sequence $b_2^2 = b_1 b_3$ and $b_1 b_4 = b_2 b_3$:

$$(aq - 3)^2 = a(aq^2 - 6), \quad a(aq^3 - 36) = (aq - 3)(aq^2 - 6)$$

These simplify to $2a(q - 1) = 3$ and $a(q^2 + 2q - 12) = 6$. Dividing: $q^2 + 2q - 12 = 4(q - 1)$, so $q = 4$ or $q = -2$, and then $a = \frac{1}{2}$ or $a = -\frac{1}{2}$. The sequences are

$$\begin{aligned} &\frac{1}{2}, 2, 8, 32 \quad \text{and} \quad \frac{1}{2}, -1, 2, -4, \quad \text{or} \\ &-\frac{1}{2}, 1, -2, 4 \quad \text{and} \quad -\frac{1}{2}, -2, -8, -32 \end{aligned}$$

5. Solve the following inequality over the real numbers.

$$\sqrt{\log_2 16 - \log_2 x^5 + (\log_2 x)^2} \leq \sqrt{2(\log_2 x)^2 - 5\log_2 x}$$

Solution: We need $x > 0$. With $y = \log_2 x$, the inequality is $\sqrt{y^2 - 5y + 4} \leq \sqrt{2y^2 - 5y}$. Both square roots are defined if $y^2 - 5y + 4 \geq 0$ and $2y^2 - 5y \geq 0$, that is, $y \leq 0$ or $y \geq 4$. Both sides are nonnegative, so squaring is allowed: $y^2 - 5y + 4 \leq 2y^2 - 5y$, that is, $y^2 \geq 4$. So $y \leq -2$ or $y \geq 4$, and the solution is

$$0 < x \leq \frac{1}{4} \quad \text{or} \quad x \geq 16$$

Note: The solution sketch in KoMaL gives $0 < x \leq \frac{1}{2}$, which is a slip: $\log_2 x \leq -2$ means $x \leq \frac{1}{4}$.

6. Solve the following equation over the real numbers.

$$\left| \frac{4\sin^2 x - 1}{\cos x} \right| = \frac{4\cos^2 x - 3}{\cos x}$$

Solution: We need $\cos x \neq 0$. Since $4\cos^2 x - 3 = 1 - 4\sin^2 x$, the equation is $|f| = -f$ with $f = \frac{4\sin^2 x - 1}{\cos x}$, which holds exactly when $f \leq 0$.

If $\cos x > 0$: $\sin^2 x \leq \frac{1}{4}$, so $-\frac{\pi}{6} + 2k\pi \leq x \leq \frac{\pi}{6} + 2k\pi$.

If $\cos x < 0$: $\sin^2 x \geq \frac{1}{4}$, so $\frac{\pi}{2} + 2k\pi < x \leq \frac{5\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi \leq x < \frac{3\pi}{2} + 2k\pi$.

7. The parallel sides of the trapezoid $ABCD$ are AB and CD , $AB = 3 \cdot CD$, $BC = AD$, and the diagonal AC is perpendicular to the diagonal BD . Find the coordinates of the vertices A and B , if $C(6, 4)$ and $D(3, 3)$.

Solution: The trapezoid is isosceles, so its diagonals meet at a point K on the axis of symmetry, and the triangle DKC is an isosceles right triangle with hypotenuse DC . With $F\left(\frac{9}{2}, \frac{7}{2}\right)$ the midpoint of DC and $\overrightarrow{DF} = \left(\frac{3}{2}, \frac{1}{2}\right)$, the vector $\overrightarrow{FK}$ is $\overrightarrow{DF}$ rotated by $\pm 90^\circ$: $\left(-\frac{1}{2}, \frac{3}{2}\right)$ or $\left(\frac{1}{2}, -\frac{3}{2}\right)$. Since $AB = 3CD$, the triangle ABK is 3 times as large as CDK (and reflected through K), so $A = C + 4\overrightarrow{CK}$ and $B = D + 4\overrightarrow{DK}$.
 $K_1 = (4, 5)$: $A(-2, 8)$, $B(7, 11)$. $K_2 = (5, 2)$: $A(2, -4)$, $B(11, -1)$.
(In both cases $\overrightarrow{AB} = (9, 3) = 3\overrightarrow{DC}$.)

8. Consider the inequality

$$((a+2)x^2 + (1-a)x + a-1)(4x - x^2 - 5) \leq 0$$

where a is a real parameter.

- a) Find the value of a for which the inequality has exactly one solution.
b) For which a does the inequality have infinitely many solutions, and for which a does it have no solution at all?

Solution: $4x - x^2 - 5 = -(x-2)^2 - 1 < 0$, so the inequality is equivalent to $g(x) = (a+2)x^2 + (1-a)x + a-1 \geq 0$.
If $a = -2$: $3x - 3 \geq 0$, infinitely many solutions.

If $a \neq -2$: the discriminant is $D = (1-a)^2 - 4(a+2)(a-1) = -3(a+3)(a-1)$. If $a+2 > 0$, then $g(x) \geq 0$ for all large x , so there are infinitely many solutions. If $a+2 < 0$, that is, $a < -2$: the solutions are the points between the roots, so there are infinitely many if $D > 0$ (that is, $-3 < a < -2$), exactly one if $D = 0$ ($a = -3$), and none if $D < 0$ ($a < -3$).

- a) $a = -3$ (the only solution is then $x = 2$).
b) Infinitely many solutions for $a > -3$; no solution for $a < -3$.