

author: Imre Rábai

In memory of Tibor Bakos ("Uncle Tibi")

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. For a deposit of 300 000 forints, a bank pays 386 400 forints after two years. In the second year the interest rate was 3% lower (three percentage points lower) than in the first year. What was the interest rate in the first year?

Solution: Let the rate in the first year be $p\%$. Then

$$300\,000 \left(1 + \frac{p}{100}\right) \left(1 + \frac{p-3}{100}\right) = 386\,400, \quad (100+p)(97+p) = 12\,880, \quad p^2 + 197p - 3180 = 0$$

so $p = 15$ (the other root, -212 , is meaningless). The rate was 15% in the first year (and 12% in the second).

2. The point P is on the line of the side AB of the rectangle $ABCD$. The line through A parallel to DP meets the line CP at Q . How far is the point Q from the line AB , if $AP = 9$, $BP = 6$, and $BC = 4.5$ units?

Solution: Since $AQ \parallel DP$, the triangles APQ and DCP are similar (AP is parallel to DC and AQ to DP , and Q, P, C are on one line). So the distance d of Q from the line AB satisfies $\frac{d}{BC} = \frac{AP}{DC} = \frac{AP}{AB}$. There are two possibilities for B : $AB = 9 + 6 = 15$ (if P is between A and B) or $AB = 9 - 6 = 3$. So

$$d = 4.5 \cdot \frac{9}{15} = 2.7 \quad \text{or} \quad d = 4.5 \cdot \frac{9}{3} = 13.5 \text{ units}$$

3. Two sides a and c of a triangle are given, and

$$\sqrt{\tan \alpha \cdot \cot \beta} = \frac{a}{b}$$

where α is the angle opposite a and β is the angle opposite b . Express the side b of the triangle in terms of a and c .

Solution: The square root is defined and positive, so $\tan \alpha$ and $\cot \beta$ have the same sign, and since a triangle has at most one obtuse angle, α and β are acute. Squaring and using the Law of Sines ($\frac{a}{b} = \frac{\sin \alpha}{\sin \beta}$):

$$\frac{\sin \alpha \cos \beta}{\cos \alpha \sin \beta} = \frac{\sin^2 \alpha}{\sin^2 \beta}, \quad \sin \beta \cos \beta = \sin \alpha \cos \alpha, \quad \sin 2\beta = \sin 2\alpha$$

So $2\alpha = 2\beta$ or $2\alpha = 180^\circ - 2\beta$, that is, $\alpha = \beta$ or $\alpha + \beta = 90^\circ$. In the first case $b = a$; in the second case the angle opposite c is a right angle, and $b = \sqrt{c^2 - a^2}$.

4. Consider the square $ABCD$ with side $2a$. The point P divides the side AB into two parts. On the segments AP and PB we draw squares so that they cover a part of the original square. How should we choose the position of P so that the uncovered area of the square is as large as possible? What is this area in terms of a ?

Solution: The uncovered area is largest when the covered area is smallest. Let $AP = a - x$ and $PB = a + x$ (with $0 \leq x < a$, choosing the labels so that $AP \leq PB$). The covered area is

$$(a - x)^2 + (a + x)^2 = 2a^2 + 2x^2 \geq 2a^2$$

with equality for $x = 0$. So P should be the midpoint of AB , and then the uncovered area is $4a^2 - 2a^2 = 2a^2$.

5. For which real numbers x is the following inequality true?

$$8|\sin x| \geq 4\cos^2 x + 1$$

Solution: Since $\cos^2 x = 1 - |\sin x|^2$, the inequality is $4|\sin x|^2 + 8|\sin x| - 5 \geq 0$, that is,

$$4\left(|\sin x| - \frac{1}{2}\right)\left(|\sin x| + \frac{5}{2}\right) \geq 0$$

The second factor is positive, so the condition is $|\sin x| \geq \frac{1}{2}$:

$$\frac{\pi}{6} + k\pi \leq x \leq \frac{5\pi}{6} + k\pi, \quad k \text{ is an integer}$$

Note: In KoMaL, the right-hand side is printed as $4\cos^2 x - 1$, but the published solution works with $4\cos^2 x + 1$, so the printed -1 is most likely a misprint. (With $4\cos^2 x - 1$ the condition would be $|\sin x| \geq \frac{\sqrt{7}-2}{2}$, which is not a nice value.)

6. Find the equation of the line that is parallel to the line $3x + 4y = 1999$ and is 2 units away from the point $C(-1, 2)$.

Solution: The line is $3x + 4y + k = 0$ for some k , and its distance from C is

$$\frac{|3 \cdot (-1) + 4 \cdot 2 + k|}{5} = \frac{|k + 5|}{5} = 2$$

so $k = 5$ or $k = -15$. The lines are $3x + 4y + 5 = 0$ and $3x + 4y - 15 = 0$.

7. Solve the following equation over the pairs of real numbers (x, y) .

$$\log_2 \left(\cos^2(xy) + \frac{1}{4\cos^2(xy)} \right) = - \left(y - \frac{1}{2} \right)^2$$

Solution: We need $\cos(xy) \neq 0$. By the inequality between the arithmetic and geometric means, $\cos^2(xy) + \frac{1}{4\cos^2(xy)} \geq 2\sqrt{\frac{1}{4}} = 1$, so the left-hand side is at least 0, while the right-hand side is at most 0. So both are 0: $y = \frac{1}{2}$ and $\cos^2(xy) = \frac{1}{4\cos^2(xy)}$, that is, $\cos^2 \frac{x}{2} = \frac{1}{2}$, so $\cos x = 2\cos^2 \frac{x}{2} - 1 = 0$. The solutions are

$$x = \frac{\pi}{2} + k\pi, \quad y = \frac{1}{2}, \quad k \text{ is an integer}$$

8. Solve the following equation over the real numbers, where a is a real parameter.

$$a \cdot 3^x + \frac{a}{3^x} = 3^{x+2} + 2a$$

Solution: With $u = 3^x > 0$ and multiplying by u : $(a - 9)u^2 - 2au + a = 0$.

If $a = 9$: $-18u + 9 = 0$, $u = \frac{1}{2}$, so $x = \log_3 \frac{1}{2} = -\log_3 2$.

If $a \neq 9$: the discriminant is $4a^2 - 4a(a - 9) = 36a$, so for $a < 0$ there is no solution. For $a > 0$:

$$u = \frac{a \pm 3\sqrt{a}}{a - 9}, \quad u_1 = \frac{\sqrt{a}}{\sqrt{a} + 3}, \quad u_2 = \frac{\sqrt{a}}{\sqrt{a} - 3}$$

u_1 is always positive, and u_2 is positive only for $a > 9$. (For $a = 0$ the equation is $-9u^2 = 0$, no solution.)

Summary: for $a \leq 0$ there is no solution; for $0 < a < 9$ the only solution is $x = \log_3 \frac{\sqrt{a}}{\sqrt{a} + 3}$; for $a = 9$ it is

$x = -\log_3 2$; for $a > 9$ there are two solutions, $x = \log_3 \frac{\sqrt{a}}{\sqrt{a} + 3}$ and $x = \log_3 \frac{\sqrt{a}}{\sqrt{a} - 3}$.