Problem Set 1

1. Solve over the real numbers:
$$\cos x + \frac{\sin^2 x}{\cos x} + \sin x + \sin 2x = \frac{1}{\cos x}$$

2. Suppose that $ABCO$ is a pyramid with a triangular base. Let us denote the vectors pointing from O to A, B, and C by $\vec{a}$, $\vec{b}$, and $\vec{c}$. Let $\vec{x}$ be the vector starting at the centroid S_B of triangle OAB and pointing to the centroid S_C of triangle OAC. Express $\vec{x}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$. Compute the ratio $S_BS_C : BC$.

3. Suppose that a, b, c, and d are consecutive integers. Prove that $a + b^2 + c^3$ is divisible by d^2.

4. The first term and common ratio of a geometric sequence are both integers. The sum of the first three terms is 21, and the sum of the nth and the previous two terms is 336. Find the sum of the first n terms in the sequence.

5. For all real numbers $0 \le x \le 5$, we define the following function:
$$f(x) = \frac{2x^2 - 9x - 11}{x^2 - 5x - 6}$$
Find the greatest and smallest value of f.

6. Solve over the real numbers:
$$4^x - 4^{\sqrt{x}+1} = 3 \cdot 2^{x+\sqrt{x}}$$

7. A sphere is inscribed in a right conical frustum. Given that the volume of the frustum is twice the volume of the sphere, what is the ratio $R : r$ where R is the radius of the base of the frustum and r is the radius of the top of the frustum?

8. A lattice point is one whose both coordinates are integers. Prove that if the center of a circle is $\left(\sqrt{5}, \frac{1}{3}\right)$ then the circle contains at most one lattice point.

Problem Set 2

9. Solve over the real numbers:

$$x + 3 \cdot \sqrt[3]{x^2} - 18 \cdot \sqrt[3]{x} = 0$$

10. We draw tangent lines to a circle from a point P outside of the circle. The line segment between P and a point of tangency is 3 units long. The line segment connecting P and the center of the circle intersects the circle in point Q. Line segment PQ is $\sqrt{3}$ units long. Compute the angle formed between the two tangent lines.

11. One vertex of an equilateral triangle is $A(-1,2)$. The center of the circumscribed circle is $P(1,4)$. Find the coordinates of the other two vertices.

12. Solve the following system of equations over the real numbers:

$$\frac{xy}{5x+4y} = 6, \qquad \frac{yz}{3y+5z} = 6, \qquad \frac{zx}{2z+3x} = 8$$

13. Suppose that $ABCD$ is a trapezoid with parallel sides $AB = a$ and $CD = b$, $a > b$. The angles at A and D are right angles. Also suppose that the bisector of angle CBA intersects side AD at its midpoint. Express the area of this trapezoid in terms of a and b.

14. Let ABC be a triangle with sides a, b, and c. We draw squares on all three sides. The vertices of the squares that are not on the triangle form a hexagon. Label the sides of the hexagon that are not on the square by x, y, and z. Prove that $x^2 + y^2 + z^2 = 3\left(a^2 + b^2 + c^2\right)$.

15. A, B, and C are cities whose distances from each other are given as $AB = 600\,\text{km}$, $BC = 800\,\text{km}$, and $AC = 800\,\text{km}$. At the same time, two planes depart. One plane travels from A to B and another from B to C. Both planes start at the same time, and follow a straight path, at the same height, with the same speed. After flying how many kilometers will be the distance between the two planes the smallest?

16. Suppose that $ABCD$ is a unit square. We write triangles into the square as follows: two vertices of the triangle are A and B, and the third vertex is on CD. Find the point P for which the perimeter of the triangle is greatest. Find the point Q for which the perimeter of the triangle is smallest.

Problem Set 3

17. The population of a town is currently 48 500. The population is increasing 7% every year. How many people lived in the town three years ago? How many percentage is the total increase during these three years?

18. Solve the following system of equations over the real numbers.

$$3^{\log_3 x} - 2^{\log_4 y} = 77, \qquad 3^{\log_3 \sqrt{x}} - 2^{\log_{16} y} = 7$$

19. A square has sides 1 unit long. On each side, we write isosceles triangles inside the square, so that the base of the triangle is the side of the square, and the angle opposite that side is 150°. Compute the exact value of the area of the square that is formed by the vertices of these triangles that are inside the square.

20. One leg of a right triangle is 1 unit long. The median drawn to the other leg is perpendicular to the median drawn to the hypotenuse. Find the lengths of the other two sides in the triangle.

21. The edges of a rectangular prism are 1, 2, and 3 units long. Find the surface area and volume of the sphere into which the prism is inscribed.

22. Find all natural numbers n for which exactly two of the following are true:
    a) $4n^2 - 360n + 8099 < 0$
    b) $n - 2$ is divisible by 7
    c) $n^2 - 2$ is divisible by 7

23. Find all values of the parameter α in the interval $[0, 2\pi]$ for which the equation

$$(2\cos\alpha - 1)\,x^2 + 4x + 4\cos\alpha + 2 = 0$$

has two real solutions, one positive and one negative.

24. How many straight lines contain the point $P\,(4,3)$ so that their intercepts are both natural numbers and the x-intercept is a prime number? Write the equation for all such lines.

Problem Set 4

25. The radius of a circle is 5. Find the values of constants A, B, and C and the coordinates of the center of the circle if its equation is
$$4x^2 + Ay^2 + Bxy + Cy - 8x - 60 = 0$$

26. One angle of a right triangle is 30°. Find the quotient of the subscribed and inscribed circle.

27. Suppose that f and g are functions, defined on $[-3\pi, 6\pi]$ as
$$f(x) = \sin\frac{x}{2} \quad \text{and} \quad g(x) = \cos\frac{x}{3}$$
How many zeroes do the fg and $f^2 + g^2$ have on the interval given?

28. Find all positive values of b for which the system
$$\begin{aligned} x^3 - y^3 &= b^2 \\ x - y &= b \end{aligned}$$
has exactly one real solution (x, y). What is that solution?

29. Find the domain of the following expressions. (The domain as a subset of $\mathbb{R}$)
$$\text{a)} \ \frac{\log\left(x^2 - 2x - 3\right)}{\log|x-3|} \qquad \text{b)} \ \frac{1}{\log(\tan|2x|)}$$

30. The vertices of a square $ABCD$ are $A(-10, 0)$, $B(-5, -10)$, $C(10, 0)$, and $D(5, 10)$. From each vertex, we drew a perpendicular line to the diagonal that does not contain the vertex. The intersection points are E, F, G, and H. Compute the exact value of the area of square $EFGH$.

31. Suppose that a, b, and c are positive numbers and are also consecutive terms in a geometric sequence. Prove that for all real numbers x,
$$\frac{1}{3} \le \frac{ax^2 + bx + c}{ax^2 - bx + c} \le 3$$
Is there a value for x for which the equation holds on one or both sides?

32. A container is shaped as a straight circular cone. It is standing on its base, and it contains water to half of its height. We turn around the container by 180° so that its circular base is upward. The height of the water is what percentage of the height of the cone now?

Problem Set 5

33. Solve over the real numbers:
$$\frac{x^2+5}{x+2}+1=\frac{x^2-4}{x-2}$$

34. The parallel sides of a trapezoid are 18 cm and 24 cm long. Another side is 15 cm long. This side and the longer parallel sides form an angle of 74.5°. Compute the fourth side of the trapezoid and its angles.

35. The product of the first three terms of a geometric sequence is 216. If we decrease the third number by 3, we obtain cosecutive elements of an arithmetic sequence. Find the three tersm in the arithmetic sequence.

36. One vertex of an equilateral triangle is $A(6,8)$. The equation of the circle inscribed into the triangle is $x^2+y^2=64$. Find the equation of the side BC and the coordinates of B and C.

37. Sketch the graph of the function $f(x)=\frac{x}{x-2}$. Where does this graph intersect the $x-$ and $y-$axis? Find the smallest and greatest values of the function on $\left[-\frac{3}{2},1\right]$ and the values of x for which these values are taken.

38. Consider a cylinder. The ratio of the lateral surface to the area of the base is 5 : 3. If we intersect the cylinder with a plane that contains the center of the cylinder, the intersection is a rectangle with a diagonal of 39 cm. Compute the surface area and volume of the cylinder.

Problem Set 6

39. Solve over the real numbers:
$$\frac{2x-1}{2x+1}=\frac{2x+1}{2x-1}+\frac{4}{1-4x^2}$$

40. A bucket is shaped a straight circular fustrum. The base of the bucket is of diameter 20 cm, the top of the bucket is of diameter 30 cm, and the length around the side is 27 cm. The bucket is filled with mortar. If we need to use $\frac{1}{4}$ of the mortar first to fix some holes, how big of a surface can be covered with the rest if we use a 6 mm thick coating? (10 mm = 1 cm).

41. The sum of the first four terms in an arithmetic sequence is 0. The sum of the squares of the same numbers is 20. Find these numbers.

42. Suppose that $ABCD$ is a parallelogram with sides $AB=5$ cm and $BC=3$ cm. Point P is on side BC, twice as far from B as from C. Lines DP and AB intersect each other in point Q. Compute the ratio between the areas of triangle DBQ and paralelogram $ABCD$.

43. Find an equation of the tangent lines drawn to the circle $(x+1)^2+(y-2)^2=20$ from the point $P(7,-4)$.

44. Sketch the graph of $f(x)=||x+1|-|x-1||$, on domain $-2\le x\le 2$. Determine where the function is increasing, decreasing, and is constant. Find its maximum and minimum values and wherre they are taken. Also, find its range. Is the function even or odd or neither?

Problem Set 7

45. We prepared 18 kg of a mixture. The components we used cost \$500 and \$300 per kilogram. If we sell the mixture for \$390 per kilogram we would lose \$380. How many kilograms of each type were used to create the mixture?

46. In square $ABCD$, we connect vertices A and B with H_1 and H_2, where H_1 and H_2 divide side CD into three equal line segments. The square is now divided into six triangles. Find the area of each of the six triangles.

47. Find the equation of the shortest chord drawn in the circle $x^2 + y^2 + 2x - 2y = 14$ among the chords that contain the point $P(1, 3)$. Find the length of the chord.

48. The first term of an arithmetic sequence is 1. The sum of the first five terms is $\frac{1}{4}$ of the sum of the next five terms. Find the first five terms of the sequence.

49. A straight pyramid has a square base with sides 8 units long. Two consecutive edge (not of the base) form an angle of $60°$. We place a plane through a diagonal of the square base, so that the plane is perpendicular to a side edge that doesn't intersect it. Find the area of the intersection of the pyramid and the plane.

50. Determine the domain and minimum value of the real-valued function

$$f(x) = \frac{\left(x^2 - 8x + 8\right)^2 - 100}{x^2 - 8x + 18}$$

Problem Set 8

51. Solve over the real numbers:

$$\sqrt{x+10} - \sqrt{x+3} = \sqrt{2x-11}$$

52. The shorter side of a parallelogram is 8 units long, the diagonals intersect each other in a $45°$ angle. Find the perimeter of the paralelogram given that its area is 40 unit^2.

53. A factory produces two types of machine parts. 73% of the factory's income is from producing part A. What will be the increase in the revenue if the factory increases production of part A by 27% and part B by 22%?

54. The sum of the first ten terms in an arithmetic sequence is 155. The product of the first and seventh term is the same as the product of the second and third terms. Find the first tem terms in the sequence.

55. Find all rectangles with integers for both side lengths where the measure of the area is twice the measure of the perimeter.