

Technical Colleges Entrance Exam - 1986

translated from: KoMaL, September 1986, pages 251-252

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These problems were given at the written mathematics entrance examination of the technical colleges in Hungary in 1986. Problems 6/a, 7/a, and 8/a were recommended for students coming from vocational secondary schools (in place of problems 6, 7, and 8).

1. Solve the given equation over the real numbers.

$$\log(4x + 2) - \log(-6x) = \log(1 - 2x)$$

Solution: We need $4x + 2 > 0$, $-6x > 0$, and $1 - 2x > 0$, that is, $-\frac{1}{2} < x < 0$. Then the equation is

$$\begin{aligned} \frac{4x + 2}{-6x} &= 1 - 2x && \text{multiply by } -6x \\ 4x + 2 &= -6x + 12x^2 \\ 0 &= 6x^2 - 5x - 1 = (6x + 1)(x - 1) \end{aligned}$$

Only $x = -\frac{1}{6}$ is in the domain. (Check: $\log \frac{4}{3} - \log 1 = \log \frac{4}{3}$.)

2. The first term of an arithmetic sequence is -210 , and its n th term is 228 . The sum of the terms between them is 45 . Write down the first n terms of the sequence.

Solution: The sum of the first n terms is $-210 + 45 + 228 = 63$, and it is also $\frac{n(-210 + 228)}{2} = 9n$. So $n = 7$, and

the common difference is $d = \frac{228 - (-210)}{6} = 73$. The terms are

$$-210, \quad -137, \quad -64, \quad 9, \quad 82, \quad 155, \quad 228$$

3. In triangle ABC , the angle at vertex A is a right angle, and one of the angles is 30° . The distance of vertex A from the center of the inscribed circle is $2\sqrt{2}$. Find the radius of the circumscribed circle.

Solution: The center I of the inscribed circle is on the bisector of the right angle, and its distance from the legs is r . So $AI = \frac{r}{\sin 45^\circ} = r\sqrt{2}$, and $r = 2$.

The angles are 90° , 60° , 30° , so if the hypotenuse is c , the legs are $\frac{c}{2}$ and $\frac{\sqrt{3}}{2}c$. In a right triangle, $r = \frac{a + b - c}{2}$, so

$$2 = \frac{1}{2} \left(\frac{c}{2} + \frac{\sqrt{3}}{2}c - c \right) = \frac{\sqrt{3} - 1}{4}c, \quad c = \frac{8}{\sqrt{3} - 1} = 4(\sqrt{3} + 1)$$

The radius of the circumscribed circle is half of the hypotenuse: $R = 2(\sqrt{3} + 1) \approx 5.46$.

4. For which real numbers x are the following expressions defined? What values do they take?

a) $\sin\left(\frac{\pi}{6} \cos(\pi x)\right)$ b) $\sqrt{\sin(\pi x)}$

Solution: a) The expression is defined for every real x . Since $\cos(\pi x)$ takes every value in $[-1, 1]$, the argument $\frac{\pi}{6} \cos(\pi x)$ takes every value in $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$, where $\sin$ is increasing. So the values are $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

b) We need $\sin(\pi x) \geq 0$, that is, $2k\pi \leq \pi x \leq (2k+1)\pi$, so $2k \leq x \leq 2k+1$ for some integer k . The values are $[0, 1]$.

5. Two sides of a triangle are 8 and 12, and the median drawn to the third side is 9. Find the area of the triangle.

Solution: Let c be the third side. By the formula for the median, $m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4}$, so $324 = 128 + 288 - c^2$ and $c^2 = 92$.

A quicker way to the area: reflect the triangle about the midpoint of side c . We get a parallelogram with sides 8 and 12 and a diagonal $2 \cdot 9 = 18$, and the triangle with sides 8, 12, 18 is half of it, just like the original triangle. By Heron's formula with $s = 19$:

$$T = \sqrt{19 \cdot 11 \cdot 7 \cdot 1} = \sqrt{1463} \approx 38.25$$

6. Solve the given inequality over the real numbers.

$$(x^2 - x + 1)^{x-2} > 1$$

Solution: The base $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ is always positive. A power of a positive base is greater than 1 exactly when either the base is greater than 1 and the exponent is positive, or the base is less than 1 and the exponent is negative.

$x^2 - x + 1 > 1$ means $x(x-1) > 0$, that is, $x < 0$ or $x > 1$; together with $x-2 > 0$: $x > 2$.

$x^2 - x + 1 < 1$ means $0 < x < 1$; together with $x-2 < 0$: $0 < x < 1$.

So the solution is $0 < x < 1$ or $x > 2$.

7. The base of a regular square pyramid is the square $ABCD$, its apex is O , and all of its edges are h units long. Find the area of the quadrilateral that the plane through the base edge AB and the midpoint F of the edge OC cuts out of the pyramid. What angle does this plane make with the base?

Solution: The height of the pyramid is $H = \sqrt{h^2 - \frac{h^2}{2}} = \frac{h}{\sqrt{2}}$. The plane intersects the face OCD in a line through F parallel to CD (since $AB \parallel CD$), so it goes through the midpoint G of OD . The cross-section is the isosceles trapezoid $ABFG$, with $AB = h$ and $GF = \frac{h}{2}$ (a midline of triangle OCD).

The height of the trapezoid: the midpoint of GF is at height $\frac{H}{2}$ above the base, and its projection onto the base is at distance $\frac{h}{4}$ from the center of the square, on the side of CD ; so it is at distance $\frac{h}{2} + \frac{h}{4} = \frac{3h}{4}$ from the line AB . So

$$m = \sqrt{\left(\frac{3h}{4}\right)^2 + \left(\frac{h}{2\sqrt{2}}\right)^2} = h\sqrt{\frac{9}{16} + \frac{2}{16}} = \frac{\sqrt{11}}{4}h$$

and the area is

$$T = \frac{h + \frac{h}{2}}{2} \cdot \frac{\sqrt{11}}{4}h = \frac{3\sqrt{11}}{16}h^2 \approx 0.622h^2$$

The angle φ between the plane and the base satisfies

$$\tan \varphi = \frac{H/2}{3h/4} = \frac{h/(2\sqrt{2})}{3h/4} = \frac{\sqrt{2}}{3}, \quad \varphi \approx 25.24^\circ$$

8. We write the consecutive positive integers on a board, starting with 1, up to a certain number. Then we erase one of the numbers. The arithmetic mean of the remaining numbers is $\frac{602}{17}$. Which number did we erase?

Solution: Suppose we wrote $1, 2, \dots, n$ and erased m . The mean of the remaining $n - 1$ numbers is $\frac{602}{17}$, so $n - 1$ is a multiple of 17: $n - 1 = 17k$, and the sum of the remaining numbers is $602k$.

Erasing a number from $1, \dots, n$ changes the mean, $\frac{n+1}{2}$, by less than 1 in either direction (the mean of the rest is between $\frac{n}{2}$ and $\frac{n+2}{2}$). Since $\frac{602}{17} \approx 35.4$, we get $n \approx 69$ or 70 , and $n - 1 = 17k$ gives $n = 69, k = 4$.

Then the sum of all numbers is $\frac{69 \cdot 70}{2} = 2415$, the sum of the remaining ones is $602 \cdot 4 = 2408$, so we erased the number 7.

- 6/a. The height of a trapezoid inscribed in a circle is 8, and one of its legs is seen from the center of the circle at an angle of 60° . Find the area of the trapezoid.

Solution: A trapezoid inscribed in a circle is isosceles. The leg BC is seen from the center at 60° , so the inscribed angle BAC that belongs to it is 30° : the diagonal AC makes an angle of 30° with the base AB .

In an isosceles trapezoid with parallel sides a and c , the projection of a diagonal onto the longer base is $\frac{a+c}{2}$. So

$$\frac{a+c}{2} = \frac{8}{\tan 30^\circ} = 8\sqrt{3}, \text{ and the area is}$$

$$T = \frac{a+c}{2} \cdot 8 = 64\sqrt{3} \approx 110.85$$

- 7/a. The line g through the origin cuts a chord of length 10 from the circle $(x-5)^2 + (y-10)^2 = 50$. Find the equation of g and the coordinates of the endpoints of the chord.

Solution: The center of the circle is $(5, 10)$ and its radius is $\sqrt{50}$. A chord of length 10 is at distance $\sqrt{50 - 25} = 5$ from the center. The line $x = 0$ works: its distance from $(5, 10)$ is 5. Otherwise the line is $y = mx$, and

$$\frac{|5m - 10|}{\sqrt{m^2 + 1}} = 5, \quad (5m - 10)^2 = 25(m^2 + 1), \quad m = \frac{3}{4}$$

So there are two such lines.

$x = 0$: $25 + (y - 10)^2 = 50$, so the endpoints are $(0, 5)$ and $(0, 15)$.

$y = \frac{3}{4}x$: $(x - 5)^2 + \left(\frac{3}{4}x - 10\right)^2 = 50$ simplifies to $x^2 - 16x + 48 = 0$, so $x = 4$ or $x = 12$, and the endpoints are $(4, 3)$ and $(12, 9)$.

8/a. Solve the given equation over the real numbers.

$$\sqrt{x+2+4\sqrt{x-2}} + \sqrt{x+7-6\sqrt{x-2}} = 5$$

Solution: Let $t = \sqrt{x-2} \geq 0$, so $x = t^2 + 2$. Then

$$x+2+4\sqrt{x-2} = t^2 + 4t + 4 = (t+2)^2, \quad x+7-6\sqrt{x-2} = t^2 - 6t + 9 = (t-3)^2$$

so the equation is $(t+2) + |t-3| = 5$, that is, $|t-3| = 3-t$. This holds exactly when $t \leq 3$. So $0 \leq \sqrt{x-2} \leq 3$, and every x with $2 \leq x \leq 11$ is a solution.

Economics Entrance Exam - 1986

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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1986. Problems 6/a, 7/a, and 8/a were recommended for students coming from vocational secondary schools (in place of problems 6, 7, and 8).

1. Find n if the ratio of the sum of the first n positive even numbers and the sum of the first n positive odd numbers is $\frac{21}{20}$.

Solution: The sum of the first n positive even numbers is $2 + 4 + \dots + 2n = n(n+1)$, and the sum of the first n positive odd numbers is $1 + 3 + \dots + (2n-1) = n^2$. So

$$\frac{n(n+1)}{n^2} = \frac{n+1}{n} = \frac{21}{20}, \quad 20n + 20 = 21n, \quad n = 20$$

2. Solve the given system of equations over the real numbers.

$$\begin{aligned} 8^{2x+1} &= 32 \cdot 2^{4y-1} \\ 5 \cdot 5^{x-y} &= \sqrt{25^{2y+1}} \end{aligned}$$

Solution: We write both equations with a common base. The first one is $2^{6x+3} = 2^5 \cdot 2^{4y-1} = 2^{4y+4}$, so $6x+3 = 4y+4$. The second one is $5^{1+x-y} = (5^{4y+2})^{\frac{1}{2}} = 5^{2y+1}$, so $1+x-y = 2y+1$, that is, $x = 3y$. Substituting, $18y+3 = 4y+4$, so

$$y = \frac{1}{14}, \quad x = \frac{3}{14}$$

3. For which real numbers are the following expressions defined?

a) $\log \log \sqrt{x}$ b) $\log \sqrt{\log x}$ c) $\sqrt{\log \log x}$

Solution: The logarithm is defined only for positive numbers.

a) We need $x > 0$ and $\log \sqrt{x} > 0$, that is, $\sqrt{x} > 1$: $x > 1$.

b) We need $\sqrt{\log x} > 0$, that is, $\log x > 0$: $x > 1$.

c) We need $\log x > 0$ and $\log \log x \geq 0$, that is, $\log x \geq 1$: $x \geq 10$.

4. In rectangle $ABCD$, $AB = 2.4 \cdot BC$. We enlarge the rectangle by a dilation with center A so that the area of the new rectangle is $\frac{9}{4}$ times the area of the original one. Because of the enlargement, the diagonal of the rectangle becomes 13 units longer. Find the sides of rectangle $ABCD$.

Solution: The ratio of the areas of similar figures is the square of the ratio of similarity, so the ratio of the dilation is $\sqrt{\frac{9}{4}} = \frac{3}{2}$. If the original diagonal is d , then $\frac{3}{2}d - d = 13$, so $d = 26$. Let $BC = b$, so $AB = 2.4b$:

$$d^2 = (2.4b)^2 + b^2 = 6.76b^2, \quad d = 2.6b = 26, \quad b = 10$$

So $BC = 10$ and $AB = 24$ units.

5. The base ABC of the triangular pyramid $OABC$ is an isosceles right triangle with $\angle ACB = 90^\circ$. The lateral edge $OC = 3\sqrt{3}$ is perpendicular to the plane of the base. The face OAB makes an angle of 60° with the plane of the base. Find the surface area and the volume of the pyramid.

Solution: Let $CA = CB = a$, and let M be the midpoint of AB . Then $CM \perp AB$, and by the theorem of three perpendiculars $OM \perp AB$ too, so the angle between the face OAB and the base is $\angle OMC = 60^\circ$. So

$$CM = \frac{OC}{\tan 60^\circ} = \frac{3\sqrt{3}}{\sqrt{3}} = 3, \quad OM = \frac{OC}{\sin 60^\circ} = 6$$

In the isosceles right triangle, $CM = \frac{a}{\sqrt{2}}$, so $a = 3\sqrt{2}$ and $AB = 6$. The volume is

$$V = \frac{1}{3} \cdot \frac{a^2}{2} \cdot OC = \frac{1}{3} \cdot 9 \cdot 3\sqrt{3} = 9\sqrt{3} \approx 15.59$$

The surface consists of the base (area 9), the face OAB (area $\frac{1}{2} \cdot 6 \cdot 6 = 18$), and the two right triangles OCA and OCB (each with area $\frac{1}{2} \cdot 3\sqrt{2} \cdot 3\sqrt{3} = \frac{9\sqrt{6}}{2}$). So

$$A = 9 + 18 + 9\sqrt{6} = 27 + 9\sqrt{6} \approx 49.05$$

6. Let $ABCD$ be a quadrilateral in the plane. Give a construction for the point P for which $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = \mathbf{0}$.

Solution: Using position vectors, $\overrightarrow{PA} = \mathbf{a} - \mathbf{p}$, etc., so the condition is $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} - 4\mathbf{p} = \mathbf{0}$, that is,

$$\mathbf{p} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4} = \frac{1}{2} \left(\frac{\mathbf{a} + \mathbf{c}}{2} + \frac{\mathbf{b} + \mathbf{d}}{2} \right)$$

Construction: find the midpoint E of the diagonal AC and the midpoint F of the diagonal BD ; then P is the midpoint of EF . (If $E = F$, as in a parallelogram, then $P = E$.)

7. How should we choose p and q in the polynomial $x^2 + px + q$ if we want to get the polynomial $x^4 + 1$ when we multiply it by a suitably chosen quadratic polynomial?

Solution: Since the product must have leading coefficient 1, the other factor is $x^2 + rx + s$. Comparing the coefficients of

$$(x^2 + px + q)(x^2 + rx + s) = x^4 + (p+r)x^3 + (q+s+pr)x^2 + (ps+qr)x + qs$$

with $x^4 + 1$: $r = -p$, $q + s = p^2$, $p(s - q) = 0$, $qs = 1$.

If $p = 0$, then $s = -q$ and $qs = -q^2 = 1$, which is impossible. So $s = q$, then $q^2 = 1$ and $2q = p^2 \geq 0$, so $q = 1$ and $p = \pm\sqrt{2}$. Indeed,

$$(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) = (x^2 + 1)^2 - 2x^2 = x^4 + 1$$

So $p = \sqrt{2}$, $q = 1$ or $p = -\sqrt{2}$, $q = 1$.

8. Solve the given equation over the set of integers.

$$\cos\left(\frac{\pi}{4}\left(3x - \sqrt{9x^2 - 16x - 80}\right)\right) = 1$$

Solution: The cosine is 1 exactly when its argument is $2k\pi$, so $3x - \sqrt{9x^2 - 16x - 80} = 8k$ for some integer k , that is, $\sqrt{9x^2 - 16x - 80} = 3x - 8k$, where we need $3x - 8k \geq 0$. Squaring:

$$9x^2 - 16x - 80 = 9x^2 - 48kx + 64k^2, \quad x(48k - 16) = 64k^2 + 80, \quad x = \frac{4k^2 + 5}{3k - 1}$$

Since $9(4k^2 + 5) = (3k - 1)(12k + 4) + 49$ and $3k - 1$ is not divisible by 3, x is an integer only if $3k - 1$ divides 49. So $3k - 1 \in \{\pm 1, \pm 7, \pm 49\}$, and k is an integer only for $3k - 1 = -1, -7, -49$, that is, $k = 0, -2, -16$.

$k = 0$: $x = -5$, but then $3x - 8k = -15 < 0$. $k = -2$: $x = -3$, and $3x - 8k = 7$. $k = -16$: $x = -21$, and $3x - 8k = 65$.

So the solutions are $x = -3$ and $x = -21$. (Check: $\sqrt{81 + 48 - 80} = 7$ and $3(-3) - 7 = -16 = 8 \cdot (-2)$; $\sqrt{3969 + 336 - 80} = 65$ and $3(-21) - 65 = -128 = 8 \cdot (-16)$.)

6/a. A parabola with an axis parallel to the y -axis, opening upward, passes through the point $A(5, 4)$ and is tangent to the x -axis. The tangent line of the parabola at A is perpendicular to the vector $\mathbf{v}(4, -1)$. Find the equation of the parabola.

Solution: Since the parabola touches the x -axis and opens upward, its equation is $y = a(x - h)^2$ with $a > 0$. The tangent line at A has normal vector $(4, -1)$, so its slope is 4. So

$$a(5 - h)^2 = 4 \quad \text{and} \quad 2a(5 - h) = 4$$

Dividing, $\frac{5 - h}{2} = 1$, so $h = 3$ and $a = 1$. The parabola is $y = (x - 3)^2$.

7/a. Prove that a triangle is a right triangle if and only if its acute angles α and β satisfy $\sin \alpha + \sin \beta = \cos \alpha + \cos \beta$.

Solution: By the sum-to-product identities,

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}, \quad \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

Since α and β are acute, $\left|\frac{\alpha - \beta}{2}\right| < 45^\circ$, so $\cos \frac{\alpha - \beta}{2} > 0$, and $0^\circ < \frac{\alpha + \beta}{2} < 90^\circ$. So the equality holds if and only if $\tan \frac{\alpha + \beta}{2} = 1$, that is, $\alpha + \beta = 90^\circ$, that is, the third angle is 90° .

8/a. Prove that every term of the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

is greater than the previous term, but less than 1.

Solution: Each term is greater than the previous one:

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$$

And each term is less than 1: a_n is the sum of n fractions, each less than $\frac{1}{n}$, so $a_n < n \cdot \frac{1}{n} = 1$.

Entrance Exam for Universities Abroad - 1986

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These problems were given in 1986 at the written mathematics entrance examination of Hungarian students applying to study at universities abroad. Problems 1–5 were the same for everyone. Of problems 6–8, the ones marked A were recommended for students who graduated from academic high schools, and the ones marked B for students who graduated from vocational secondary schools.

1. Solve the given equation over the real numbers.

$$\log(36^{5x} + 792) = \log 16 + \log 63$$

Solution: The right-hand side is $\log(16 \cdot 63) = \log 1008$, and the logarithm is one-to-one, so

$$36^{5x} + 792 = 1008, \quad 36^{5x} = 216, \quad 6^{10x} = 6^3, \quad x = \frac{3}{10}$$

2. In a right triangle, the bisector of one of the acute angles divides the opposite side into two segments of length 1 and 2. Find the angles and the sides of the triangle.

Solution: Let the right angle be at C , and let the bisector of the angle α at A meet the leg BC at D . By the angle bisector theorem, $\frac{CD}{DB} = \frac{AC}{AB}$. Since the hypotenuse AB is longer than the leg AC , CD is the shorter segment: $CD = 1$, $DB = 2$, and $\frac{AC}{AB} = \frac{1}{2}$. So $\cos \alpha = \frac{1}{2}$, $\alpha = 60^\circ$, and the angle at B is 30° .

Then $BC = 3$, $AC = \frac{BC}{\tan 60^\circ} = \sqrt{3}$, and $AB = 2\sqrt{3}$. The angles are 90° , 60° , 30° , and the sides are $\sqrt{3}$, 3 , $2\sqrt{3}$.

3. Solve the given system of equations over the real numbers.

$$\sqrt{x + 3y + 1} = 2, \quad \sqrt{2x - y + 2} = 7y - 6$$

Solution: From the first equation, $x + 3y + 1 = 4$, so $x = 3 - 3y$. Then $2x - y + 2 = 8 - 7y$, and the second equation is $\sqrt{8 - 7y} = 7y - 6$, where we need $7y - 6 \geq 0$. Squaring:

$$8 - 7y = 49y^2 - 84y + 36, \quad 7y^2 - 11y + 4 = 0, \quad y = 1 \text{ or } y = \frac{4}{7}$$

Since $7 \cdot \frac{4}{7} - 6 < 0$, only $y = 1$ works, and then $x = 0$. (Check: $\sqrt{4} = 2$ and $\sqrt{1} = 1 = 7 - 6$.)

4. Three vertices of parallelogram $ABCD$ are $A(-3, -2)$, $B(2, 0)$, and $C(3, 3)$. The line through B perpendicular to the diagonal AC intersects the line CD at E . Find the coordinates of E .

Solution: The fourth vertex is $D = A + C - B = (-2, 1)$. The vector $\overrightarrow{AC} = (6, 5)$ is a normal vector of the line through B perpendicular to AC , so this line is $6(x - 2) + 5y = 0$, that is, $6x + 5y = 12$. The line CD has direction $\overrightarrow{DC} = (5, 2)$, so its equation is $2x - 5y = -9$. Adding the two equations, $8x = 3$, so

$$E = \left(\frac{3}{8}, \frac{39}{20} \right)$$

5. Of four numbers, the first three are consecutive terms of a geometric sequence, and the last three are consecutive terms of an arithmetic sequence. The sum of the first and the fourth number is 14, and the sum of the second and the third is 12. Find the four numbers.

Solution: Let the numbers be a, b, c, d . Then $c = 12 - b$, and since b, c, d is arithmetic, $d = 2c - b = 24 - 3b$. Since a, b, c is geometric, $a = \frac{b^2}{c} = \frac{b^2}{12 - b}$. From $a + d = 14$:

$$\begin{aligned}\frac{b^2}{12 - b} &= 3b - 10 \\ b^2 &= 46b - 3b^2 - 120 \\ 0 &= 2b^2 - 23b + 60 \\ b &= \frac{23 \pm 7}{4}\end{aligned}$$

If $b = 4$: the numbers are 2, 4, 8, 12. If $b = \frac{15}{2}$: the numbers are $\frac{25}{2}, \frac{15}{2}, \frac{9}{2}, \frac{3}{2}$.

- 6A. Solve the given system of inequalities over the real numbers.

$$x^2 - 6x - 7 < 0, \quad \sin x > 0$$

Solution: The first inequality is $(x - 7)(x + 1) < 0$, that is, $-1 < x < 7$. On this interval, $\sin x > 0$ for $0 < x < \pi$ and for $2\pi < x < 7$ (since $2\pi \approx 6.28 < 7 < 3\pi$), but not for $-1 < x \leq 0$. So the solution is

$$0 < x < \pi \quad \text{or} \quad 2\pi < x < 7$$

- 7A. Let $ABCD$ and $EFGH$ be two parallelograms in the same plane. Prove that the midpoints of the segments AE, BF, CG , and DH , if they are not on one line, are the vertices of a parallelogram.

Solution: Use position vectors, and let K, L, M, N be the midpoints of AE, BF, CG, DH . Then $\mathbf{k} = \frac{\mathbf{a} + \mathbf{e}}{2}$, etc. In a parallelogram the diagonals bisect each other, so $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$ and $\mathbf{e} + \mathbf{g} = \mathbf{f} + \mathbf{h}$. So

$$\frac{\mathbf{k} + \mathbf{m}}{2} = \frac{\mathbf{a} + \mathbf{c} + \mathbf{e} + \mathbf{g}}{4} = \frac{\mathbf{b} + \mathbf{d} + \mathbf{f} + \mathbf{h}}{4} = \frac{\mathbf{l} + \mathbf{n}}{2}$$

So the segments KM and LN have the same midpoint, that is, the diagonals of the quadrilateral $KLMN$ bisect each other, and (if the points are not on one line) $KLMN$ is a parallelogram.

- 8A. Prove the following statement for every positive integer n : if a, b, c are integers such that n divides $a + b + c$ and $a^2 + b^2 + c^2$, then n also divides $a^4 + b^4 + c^4$.

Solution: Let $s = a + b + c, t = ab + bc + ca, u = abc$, and $p = a^2 + b^2 + c^2$. Then $p = s^2 - 2t$, so $2t = s^2 - p$ is divisible by n . Also,

$$a^2b^2 + b^2c^2 + c^2a^2 = t^2 - 2su$$

(expand t^2), and so

$$a^4 + b^4 + c^4 = p^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) = p^2 - 2t^2 + 4su = p^2 - (2t)t + 4su$$

Each of the three terms is divisible by n (since n divides $p, 2t$, and s), so n divides $a^4 + b^4 + c^4$.

- 6B. In the isosceles triangle ABC , the base BC is 10 units long, the altitude from A is 12, and D is the midpoint of the base. Let M be a point of the leg AC . Find the length of the segment AM if the quadrilateral $AMDB$ is cyclic.

Solution: The altitude from A goes to the midpoint D , so $AD = 12$, $DC = 5$, and $AC = 13$. The points A, M, D, B are on a circle, and C is outside it, on the lines CMA and CDB . By the theorem on secants from an external point,

$$CM \cdot CA = CD \cdot CB, \quad CM \cdot 13 = 5 \cdot 10, \quad CM = \frac{50}{13}$$

$$\text{So } AM = 13 - \frac{50}{13} = \frac{119}{13} \approx 9.15.$$

- 7B. Solve the given system of equations over the real numbers.

$$\frac{xy}{x+y} = 1-z, \quad \frac{yz}{y+z} = 2-x, \quad \frac{zx}{z+x} = 2-y$$

Solution: The denominators cannot be 0. Multiplying out, the three equations are

$$xy = (1-z)(x+y), \quad yz = (2-x)(y+z), \quad zx = (2-y)(z+x)$$

Let $P = xy + yz + zx$. Rearranging, the three equations say

$$P = x + y, \quad P = 2y + 2z, \quad P = 2z + 2x$$

From the last two, $x = y$. Then $x + y = 2x = 2z + 2x$ gives $z = 0$, and $P = x^2 = 2x$, so $x = 0$ or $x = 2$. If $x = 0$, then $x + y = 0$, which is not allowed. So the only solution is

$$x = 2, \quad y = 2, \quad z = 0$$

$$(\text{Check: } \frac{4}{4} = 1 - 0, \quad \frac{0}{2} = 2 - 2, \quad \frac{0}{2} = 2 - 2.)$$

- 8B. Prove that if $0 \leq x < \frac{\pi}{2}$, then

$$\frac{\sin x}{1 + \tan x} \leq \frac{\sqrt{2}}{4}$$

Solution: Multiplying the numerator and the denominator by $\cos x > 0$,

$$\frac{\sin x}{1 + \tan x} = \frac{\sin x \cos x}{\sin x + \cos x}$$

Let $s = \sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$; for $0 \leq x < \frac{\pi}{2}$, $1 \leq s \leq \sqrt{2}$. Since $s^2 = 1 + 2 \sin x \cos x$, $\sin x \cos x = \frac{s^2 - 1}{2}$, and

$$\frac{\sin x \cos x}{\sin x + \cos x} = \frac{s^2 - 1}{2s} = \frac{1}{2} \left(s - \frac{1}{s} \right)$$

This is increasing in s , so it is at most $\frac{1}{2} \left(\sqrt{2} - \frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2}}{4}$, with equality at $x = \frac{\pi}{4}$.