

Entrance Exam for Universities Abroad - 1987

translated from: KoMaL, September 1987, page 250

Pdf file: KoMaL, September 1987, page 250

These problems were given in 1987 at the written mathematics entrance examination of Hungarian students applying to study at universities abroad. Problems 1–5 were the same for everyone. Of problems 6–8, the ones marked A were recommended for students who graduated from academic high schools, and the ones marked B for students who graduated from vocational secondary schools.

1. The numerators of two fractions are both 3, and the sum of their denominators is 13. The sum of the two fractions is $\frac{39}{40}$. Find the two fractions.
2. Solve the given system of equations over the real numbers.

$$7^{x+1} \cdot 2^y = 4, \quad \frac{1}{x-1} - \frac{1}{y} = \frac{4}{y(x-1)}$$

3. The diagonals of an isosceles trapezoid are perpendicular, and they divide each other into two parts in a 1 : 7 ratio. Find the area of the trapezoid if its perimeter is $180\sqrt{2}$.
4. Two circles with radii R and r ($R > r$) are tangent to each other from outside. Express, as a function of the radii, the length of the segment of their common internal tangent line that lies between the two common external tangent lines.
5. The sum of the first, third, and fifth terms of a geometric sequence is 182, and the sum of the second and fourth terms is 20 times the common ratio. Write down the first five terms of the sequence.
- 6A. Let ABC and DEF be any two triangles, and let S and T be their centroids. Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 3\overrightarrow{ST}$.
- 7A. Solve the given inequality over the real numbers.

$$\log_{|x-1|} x \geq 1$$

- 8A. Is there a line in the plane all of whose points $P(x, y)$ satisfy the inequality $x^4 - 2x^2y + y^2 - 1 \leq 0$?
- 6B. In trapezoid $ABCD$, the base AB is twice as long as the parallel side CD and twice as long as the leg AD . The other leg BC is 5 units long, and the diagonal AC is 12 units long. Find the area of the trapezoid.
- 7B. Find the solutions of the following inequalities in the interval $\left[\frac{\pi}{4}, \pi\right]$.
 - a) $\frac{1}{\sin^2 x} \leq \frac{1}{\sin 2x - \cos^2 x}$
 - b) $\log_2 \sin 2x < -1$
- 8B. Let P be a point on the circle circumscribed around the equilateral triangle ABC . Prove that the sum $PA^2 + PB^2 + PC^2$ is constant, and find its value.

Technical Universities Entrance Exam - 1987

translated from: KoMaL, October 1987, pages 291-292

Pdf file: KoMaL, October 1987, pages 291-292

These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. If the sum of 101 consecutive odd numbers is 12 827, what are the smallest and the largest of them? (9 points)
2. Solve the given system of equations over the real numbers.

$$x^2 - y^2 = \frac{5}{2}, \quad xy = \frac{\sqrt{6}}{2}$$

(10 points)

3. The reflection of the origin O about the point $A(5, 5)$ is O_1 , the reflection of O_1 about the point B is O_2 , and the reflection of O_2 about the point $C(1, 7)$ is O again. Find the coordinates of B , and prove that the quadrilateral $OABC$ is a rhombus. (13 points)
4. In the acute triangle ABC , side AB is 16 units long, and the altitude belonging to it is 12. We inscribe an isosceles right triangle in triangle ABC so that its hypotenuse is parallel to AB , its right-angle vertex is on AB , and its other two vertices are on AC and BC . Find the sides of the inscribed triangle. (13 points)
5. Solve the given equation over the real numbers.

$$\sin x + \cos x = \sin 2x - \frac{1}{2}$$

(13 points)

Recommended for students from academic high schools

6. For which x do the following expressions take positive values?
a) $2^x + 2^{-x} - \frac{17}{4}$ b) $2 + \log_{1/3}(5x - 1)$ (13 points)
7. In rectangle $ABCD$, side AB is three times side BC . The distances of an inside point P from the vertices B, A, D are $PB = 4\sqrt{2}$, $PA = \sqrt{2}$, and $PD = 2$. Find the area of the rectangle. (14 points)
8. Prove that in every triangle, the midpoint of the segment connecting the center of an excircle (a circle tangent to the side lines from outside) and the center of the inscribed circle lies on the circumscribed circle. (15 points)

Recommended for students from vocational secondary schools

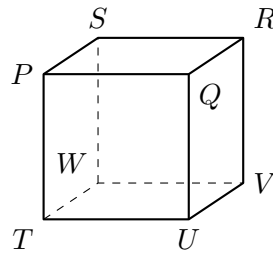
6. Prove that if the difference of the squares of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{c^2}{a^2}$, then

$$b^4 - c^4 = 4ab^2c$$

(13 points)

Problem Sets from 1987

7. The vertices of a cube are labeled as in the figure. Let A be the center of face $TUVW$, and B the center of face $UVRQ$. The edge UV intersects the plane PAB at X . Find the value of the ratio $\frac{UX}{XV}$.



(14 points)

8. The real numbers x, y, z satisfy

$$x + y = z - 1, \quad xy = z^2 - 7z + 10$$

For which values of x, y, z is the sum $x^2 + y^2$ maximal?

(15 points)

Economics Entrance Exam - 1987

translated from: KoMaL, November 1987, pages 361-362

Pdf file: KoMaL, November 1987, pages 361-362

These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

- Find the exact value of the given expression without using a calculator or tables.

$$\frac{5 \cdot 8^{-\frac{2}{3}} + \frac{2}{\sqrt{3}} \left(\sin \frac{16\pi}{3} \right)^3}{\log_4 2}$$

(9 points)

- The vertices A and B of parallelogram $ABCD$ are $A(1, 4)$ and $B(6, 6)$. The point $P(10, 18)$ is on the line BC , and the point $R(-1, 11)$ is on the line CD . Find the perimeter of the parallelogram. (11 points)
- The base of an isosceles triangle is 36 units long, and its legs are 30 units long. How far is the centroid of the triangle from the center of its circumscribed circle? (12 points)
- Find the largest subset of the real numbers on which the following expressions are defined.

a) $\frac{\sqrt{x^2 + 4x - 5}}{|x + 2| - 3}$

(8 points)

b) $\log(1 - \sin 2x)$

(5 points)

- The base of a triangular pyramid is an equilateral triangle, and its lateral faces are congruent isosceles triangles. The cross-section through one lateral edge, perpendicular to the opposite base edge, has an area of 150 cm^2 . The volume of the pyramid is 1500 cm^3 . What angle does each lateral face make with the base? (13 points)

Recommended for students from academic high schools

- Sketch the set of points $P(x, y)$ in the coordinate plane whose coordinates satisfy the following conditions.

a) $|x| \leq |y|$

(5 points)

b) $|x + 1| + |y + 1| \leq 2$

(8 points)

- Solve the given system of equations over the real numbers.

$$x^{\log_8 y} + y^{\log_8 x} = 4$$

$$\log_4 x - \log_4 y = 1$$

(14 points)

- Let O be a point in the plane, and let $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ be unit vectors such that O is outside the triangle ABC . Prove that

$$|\vec{OA} + \vec{OB} + \vec{OC}| > 1$$

(15 points)

Recommended for students from vocational secondary schools

6. Solve the given inequality over the real numbers.

$$2 \log_x \frac{1}{4} + \log_4 \frac{1}{x} \leq -3$$

(13 points)

7. In triangle ABC , the median AA_1 is perpendicular to AB . Let a, b, c be the lengths of the sides BC, CA, AB . Express the cosine of the angle CAB in terms of b and c , and the cosine of the angle BCA in terms of a and b . (14 points)

8. Let a and b be real numbers with $a > b$ and $ab = 1$. Prove that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2}$$

(15 points)