

Entrance Exam for Universities Abroad - 1987

translated from: KoMaL, September 1987, page 250

Pdf file: KoMaL, September 1987, page 250

These problems were given in 1987 at the written mathematics entrance examination of Hungarian students applying to study at universities abroad. Problems 1–5 were the same for everyone. Of problems 6–8, the ones marked A were recommended for students who graduated from academic high schools, and the ones marked B for students who graduated from vocational secondary schools.

1. The numerators of two fractions are both 3, and the sum of their denominators is 13. The sum of the two fractions is $\frac{39}{40}$. Find the two fractions.

Solution: Let the denominators be p and q . Then $p + q = 13$ and

$$\frac{3}{p} + \frac{3}{q} = \frac{3(p+q)}{pq} = \frac{39}{pq} = \frac{39}{40}, \quad \text{so} \quad pq = 40$$

So p and q are the roots of $t^2 - 13t + 40 = 0$, that is, 5 and 8. The fractions are $\frac{3}{5}$ and $\frac{3}{8}$.

2. Solve the given system of equations over the real numbers.

$$7^{x+1} \cdot 2^y = 4, \quad \frac{1}{x-1} - \frac{1}{y} = \frac{4}{y(x-1)}$$

Solution: We need $x \neq 1$ and $y \neq 0$. Multiplying the second equation by $y(x-1)$: $y - (x-1) = 4$, so $y = x + 3$. Substituting into the first equation,

$$7^{x+1} \cdot 2^{x+3} = 4, \quad 7^{x+1} \cdot 2^{x+1} \cdot 4 = 4, \quad 14^{x+1} = 1$$

So $x = -1$ and $y = 2$.

3. The diagonals of an isosceles trapezoid are perpendicular, and they divide each other into two parts in a 1 : 7 ratio. Find the area of the trapezoid if its perimeter is $180\sqrt{2}$.

Solution: Let M be the intersection of the diagonals. By symmetry, the diagonals are equal and M divides both in the same ratio, so $MA = MB = 7k$ and $MC = MD = k$ for some k . Since the diagonals are perpendicular, the triangles around M are right triangles, so

$$AB = 7\sqrt{2}k, \quad CD = \sqrt{2}k, \quad AD = BC = \sqrt{49k^2 + k^2} = 5\sqrt{2}k$$

The perimeter is $18\sqrt{2}k = 180\sqrt{2}$, so $k = 10$, and each diagonal is $8k = 80$. The area of a quadrilateral with perpendicular diagonals is half of their product:

$$T = \frac{80 \cdot 80}{2} = 3200 \text{ square units.}$$

4. Two circles with radii R and r ($R > r$) are tangent to each other from outside. Express, as a function of the radii, the length of the segment of their common internal tangent line that lies between the two common external tangent lines.

Solution: Let A be the point of tangency of the circles, and let one external tangent line touch the circles at E_1 and E_2 . The internal tangent line at A meets E_1E_2 at a point M . The tangent segments from M are equal: $ME_1 = MA = ME_2$, so $MA = \frac{E_1E_2}{2}$.

The length of the external tangent segment: the centers are $R + r$ apart, and the radii to E_1 and E_2 are perpendicular to E_1E_2 , so

$$E_1E_2 = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$$

So $MA = \sqrt{Rr}$. The same holds for the other external tangent line, on the other side of A . So the segment of the internal tangent between the two external tangents has length $2\sqrt{Rr}$.

5. The sum of the first, third, and fifth terms of a geometric sequence is 182, and the sum of the second and fourth terms is 20 times the common ratio. Write down the first five terms of the sequence.

Solution: Let the first term be a and the ratio q (with $a \neq 0, q \neq 0$). Then

$$a(1 + q^2 + q^4) = 182, \quad aq(1 + q^2) = 20q, \text{ so } a(1 + q^2) = 20$$

Dividing, with $u = q^2$:

$$\frac{1 + u + u^2}{1 + u} = \frac{182}{20}, \quad 10u^2 - 81u - 81 = 0, \quad u = 9 \text{ (or } u = -\frac{9}{10}, \text{ which is impossible)}$$

So $q = \pm 3$ and $a = \frac{20}{10} = 2$. The sequence is

$$2, 6, 18, 54, 162 \quad \text{or} \quad 2, -6, 18, -54, 162$$

- 6A. Let ABC and DEF be any two triangles, and let S and T be their centroids. Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 3\overrightarrow{ST}$.

Solution: With position vectors, $\mathbf{s} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$ and $\mathbf{t} = \frac{\mathbf{d} + \mathbf{e} + \mathbf{f}}{3}$, so

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = (\mathbf{d} + \mathbf{e} + \mathbf{f}) - (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 3\mathbf{t} - 3\mathbf{s} = 3\overrightarrow{ST}$$

- 7A. Solve the given inequality over the real numbers.

$$\log_{|x-1|} x \geq 1$$

Solution: We need $x > 0$, $|x-1| > 0$, and $|x-1| \neq 1$, that is, $x > 0$, $x \neq 1$, $x \neq 2$. We write $1 = \log_{|x-1|} |x-1|$.

If the base is greater than 1, that is, $x > 2$: we need $x \geq |x-1| = x-1$, which is always true. So every $x > 2$ is a solution.

If the base is between 0 and 1, that is, $0 < x < 2$ and $x \neq 1$: the inequality reverses, $x \leq |x-1|$. For $0 < x < 1$ this means $x \leq 1-x$, so $0 < x \leq \frac{1}{2}$; for $1 < x < 2$ it means $x \leq x-1$, which is false.

So the solution is $0 < x \leq \frac{1}{2}$ or $x > 2$.

8A. Is there a line in the plane all of whose points $P(x, y)$ satisfy the inequality $x^4 - 2x^2y + y^2 - 1 \leq 0$?

Solution: No. The inequality is $(y - x^2)^2 \leq 1$, that is, $x^2 - 1 \leq y \leq x^2 + 1$: the region between two parabolas. A vertical line $x = c$ contains points with y arbitrarily large, which are not in the region. A non-vertical line $y = mx + k$ is not in the region either, since $x^2 - 1 - (mx + k)$ is positive for large x , so the points of the line with large x are below the lower parabola.

6B. In trapezoid $ABCD$, the base AB is twice as long as the parallel side CD and twice as long as the leg AD . The other leg BC is 5 units long, and the diagonal AC is 12 units long. Find the area of the trapezoid.

Solution: Let $CD = AD = a$, so $AB = 2a$, and let F be the midpoint of AB . Then $AF = a = DC$ and $AF \parallel DC$, so $AFCD$ is a parallelogram, and since $AD = a$, it is a rhombus: $FC = a$. So F is at distance a from A , B , and C , and by the converse of Thales' theorem, the angle at C in triangle ABC is a right angle. So

$$AB = \sqrt{12^2 + 5^2} = 13, \quad a = 6.5, \quad \text{and the height is } m = \frac{AC \cdot BC}{AB} = \frac{60}{13}$$

The area is

$$T = \frac{13 + 6.5}{2} \cdot \frac{60}{13} = 45 \text{ square units.}$$

7B. Find the solutions of the following inequalities in the interval $\left[\frac{\pi}{4}, \pi\right]$.

a) $\frac{1}{\sin^2 x} \leq \frac{1}{\sin 2x - \cos^2 x}$ b) $\log_2 \sin 2x < -1$

Solution: a) We need $\sin x \neq 0$ and $D = \sin 2x - \cos^2 x \neq 0$. The left-hand side is positive. If $D < 0$, the right-hand side is negative, and the inequality is false. If $D > 0$, the inequality is equivalent to $D \leq \sin^2 x$, that is,

$$0 \leq \sin^2 x - 2 \sin x \cos x + \cos^2 x = (\sin x - \cos x)^2$$

which is always true. So the solutions are exactly the x with $D = \cos x (2 \sin x - \cos x) > 0$. In $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x > 0$ and $2 \sin x > \cos x$, so $D > 0$; at $\frac{\pi}{2}$, $D = 0$; in $\left(\frac{\pi}{2}, \pi\right]$, $\cos x < 0 < 2 \sin x - \cos x$ (or $\sin x = 0$ at π), so $D < 0$. The solution is $\frac{\pi}{4} \leq x < \frac{\pi}{2}$.

b) We need $0 < \sin 2x < \frac{1}{2}$. For $x \in \left[\frac{\pi}{4}, \pi\right]$, $2x \in \left[\frac{\pi}{2}, 2\pi\right]$, and $0 < \sin 2x < \frac{1}{2}$ holds for $\frac{5\pi}{6} < 2x < \pi$. The solution is $\frac{5\pi}{12} < x < \frac{\pi}{2}$.

8B. Let P be a point on the circle circumscribed around the equilateral triangle ABC . Prove that the sum $PA^2 + PB^2 + PC^2$ is constant, and find its value.

Solution: Let O be the center and R the radius of the circle, and use position vectors from O . Then $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = |\mathbf{p}| = R$, and since O is also the centroid, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. So

$$PA^2 + PB^2 + PC^2 = \sum (\mathbf{p} - \mathbf{a})^2 = 3R^2 + 3R^2 - 2\mathbf{p} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 6R^2$$

which is constant. If the side of the triangle is a , then $R = \frac{a}{\sqrt{3}}$, and the constant is $6R^2 = 2a^2$.

Technical Universities Entrance Exam - 1987

translated from: KoMaL, October 1987, pages 291-292

Pdf file: KoMaL, October 1987, pages 291-292

These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. If the sum of 101 consecutive odd numbers is 12 827, what are the smallest and the largest of them? (9 points)

Solution: The numbers form an arithmetic sequence with 101 terms, so their average is the middle (51st) term: $\frac{12\,827}{101} = 127$. There are 50 terms on each side, each 2 apart, so the smallest is $127 - 100 = 27$ and the largest is $127 + 100 = 227$.

2. Solve the given system of equations over the real numbers.

$$x^2 - y^2 = \frac{5}{2}, \quad xy = \frac{\sqrt{6}}{2}$$

(10 points)

Solution: Let $u = x^2$ and $v = -y^2$. Then $u + v = \frac{5}{2}$ and $uv = -(xy)^2 = -\frac{3}{2}$, so u and v are the roots of $t^2 - \frac{5}{2}t - \frac{3}{2} = 0$, that is, $2t^2 - 5t - 3 = 0$: $t = 3$ and $t = -\frac{1}{2}$. Since $u \geq 0 \geq v$, $x^2 = 3$ and $y^2 = \frac{1}{2}$. Since $xy > 0$, x and y have the same sign, so

$$x = \sqrt{3}, y = \frac{\sqrt{2}}{2} \quad \text{or} \quad x = -\sqrt{3}, y = -\frac{\sqrt{2}}{2}$$

3. The reflection of the origin O about the point $A(5, 5)$ is O_1 , the reflection of O_1 about the point B is O_2 , and the reflection of O_2 about the point $C(1, 7)$ is O again. Find the coordinates of B , and prove that the quadrilateral $OABC$ is a rhombus. (13 points)

Solution: The reflection about a point M takes X to $2M - X$. So $O_1 = (10, 10)$, and since the reflection of O_2 about C is O , $O_2 = 2C - O = (2, 14)$. B is the midpoint of O_1O_2 : $B = (6, 12)$.

Now $\overrightarrow{OA} = (5, 5) = \overrightarrow{CB}$, so $OABC$ is a parallelogram, and $|\overrightarrow{OA}| = \sqrt{50} = |\overrightarrow{OC}|$, so its sides are equal: it is a rhombus.

4. In the acute triangle ABC , side AB is 16 units long, and the altitude belonging to it is 12. We inscribe an isosceles right triangle in triangle ABC so that its hypotenuse is parallel to AB , its right-angle vertex is on AB , and its other two vertices are on AC and BC . Find the sides of the inscribed triangle. (13 points)

Solution: Let the hypotenuse of the inscribed triangle be L , at distance d from AB . The right-angle vertex is on AB , and the height of an isosceles right triangle belonging to the hypotenuse is half of the hypotenuse, so $d = \frac{L}{2}$. The hypotenuse cuts off from ABC a triangle similar to it, with height $12 - d$, so

$$\frac{L}{16} = \frac{12 - d}{12} = \frac{12 - L/2}{12}, \quad 12L = 192 - 8L, \quad L = \frac{48}{5} = 9.6$$

The legs are $\frac{L}{\sqrt{2}} = \frac{24\sqrt{2}}{5} \approx 6.79$. So the sides of the inscribed triangle are 9.6, $\frac{24\sqrt{2}}{5}$, and $\frac{24\sqrt{2}}{5}$.

5. Solve the given equation over the real numbers.

$$\sin x + \cos x = \sin 2x - \frac{1}{2}$$

(13 points)

Solution: Let $s = \sin x + \cos x$. Then $s^2 = 1 + \sin 2x$, so $\sin 2x = s^2 - 1$, and

$$s = s^2 - \frac{3}{2}, \quad 2s^2 - 2s - 3 = 0, \quad s = \frac{1 \pm \sqrt{7}}{2}$$

Since $s = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$, we need $|s| \leq \sqrt{2}$, and $\frac{1 + \sqrt{7}}{2} \approx 1.82$ is too large. So

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1 - \sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14} - \sqrt{2}}{4} \approx -0.5819$$

Let $\beta = \arcsin \frac{\sqrt{14} - \sqrt{2}}{4} \approx 35.58^\circ$. Then $x + 45^\circ = -\beta + k \cdot 360^\circ$ or $x + 45^\circ = 180^\circ + \beta + k \cdot 360^\circ$, so

$$x \approx -80.58^\circ + k \cdot 360^\circ \quad \text{or} \quad x \approx 170.58^\circ + k \cdot 360^\circ$$

Recommended for students from academic high schools

6. For which x do the following expressions take positive values?

a) $2^x + 2^{-x} - \frac{17}{4}$ b) $2 + \log_{1/3}(5x - 1)$ (13 points)

Solution: a) Let $t = 2^x > 0$. Then $t + \frac{1}{t} > \frac{17}{4}$, that is, $4t^2 - 17t + 4 = (4t - 1)(t - 4) > 0$, so $t < \frac{1}{4}$ or $t > 4$. So $x < -2$ or $x > 2$.

b) We need $5x - 1 > 0$. The inequality is $\log_{1/3}(5x - 1) > -2 = \log_{1/3} 9$, and since the base is less than 1, $5x - 1 < 9$. So $\frac{1}{5} < x < 2$.

7. In rectangle $ABCD$, side AB is three times side BC . The distances of an inside point P from the vertices B, A, D are $PB = 4\sqrt{2}$, $PA = \sqrt{2}$, and $PD = 2$. Find the area of the rectangle. (14 points)

Solution: Use coordinates: $A(0, 0)$, $B(3a, 0)$, $D(0, a)$, and $P(x, y)$. Then

$$x^2 + y^2 = 2, \quad (x - 3a)^2 + y^2 = 32, \quad x^2 + (y - a)^2 = 4$$

Subtracting the first equation from the other two: $-6ax + 9a^2 = 30$ and $-2ay + a^2 = 2$, so

$$x = \frac{9a^2 - 30}{6a}, \quad y = \frac{a^2 - 2}{2a}$$

Substituting into $x^2 + y^2 = 2$ and writing $s = a^2$: $(9s - 30)^2 + 9(s - 2)^2 = 72s$, that is, $5s^2 - 36s + 52 = 0$, so $s = 2$ or $s = \frac{26}{5}$. If $s = 2$, then $x < 0$, and P would be outside the rectangle. So $a^2 = \frac{26}{5}$, and the area is

$$3a^2 = \frac{78}{5} = 15.6 \text{ square units.}$$

8. Prove that in every triangle, the midpoint of the segment connecting the center of an excircle (a circle tangent to the side lines from outside) and the center of the inscribed circle lies on the circumscribed circle. (15 points)

Solution: Let I be the center of the inscribed circle and I_a the center of the excircle opposite A . Both are on the bisector of the angle at A . Let this bisector meet the circumscribed circle again at M . We show that M is the midpoint of II_a . Let α, β, γ be the angles of the triangle.

$MB = MI$: $\angle MBC = \angle MAC = \frac{\alpha}{2}$ (inscribed angles on the arc MC), so $\angle MBI = \frac{\alpha}{2} + \frac{\beta}{2}$. The angle $\angle MIB$ is an exterior angle of triangle ABI , so it is $\frac{\alpha}{2} + \frac{\beta}{2}$ too. So triangle MBI is isosceles, and $MB = MI$.

On the other hand, BI and BI_a are the interior and exterior bisectors at B , so $\angle IBI_a = 90^\circ$, and B is on the circle with diameter II_a . So the midpoint N of II_a satisfies $NB = NI$.

Both M and N are on the line AI , and both are on the perpendicular bisector of BI . This perpendicular bisector meets the line AI in only one point, so $M = N$. So the midpoint of II_a is M , which is on the circumscribed circle. (The same argument works for the other two excircles.)

Recommended for students from vocational secondary schools

6. Prove that if the difference of the squares of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{c^2}{a^2}$, then

$$b^4 - c^4 = 4ab^2c$$

(13 points)

Solution: The equation as printed in the original is $ax^2 + bx - c = 0$. With that equation, the same computation gives $b^4 - c^4 = -4ab^2c$, so the minus sign is evidently a misprint, and we prove the statement for $ax^2 + bx + c = 0$.

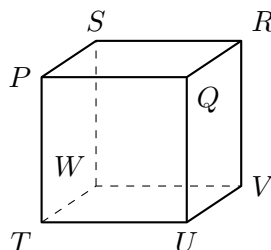
Let the roots be x_1 and x_2 . By Vieta's formulas, $x_1 + x_2 = -\frac{b}{a}$ and $x_1x_2 = -\frac{c}{a}$. The condition is

$$x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) = -\frac{b}{a}(x_1 - x_2) = \frac{c^2}{a^2}, \quad \text{so} \quad x_1 - x_2 = -\frac{c^2}{ab}$$

Squaring, and using $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$:

$$\frac{c^4}{a^2b^2} = \frac{b^2}{a^2} - \frac{4c}{a}, \quad c^4 = b^4 - 4ab^2c, \quad b^4 - c^4 = 4ab^2c$$

7. The vertices of a cube are labeled as in the figure. Let A be the center of face $TUVW$, and B the center of face $UVRQ$. The edge UV intersects the plane PAB at X . Find the value of the ratio $\frac{UX}{XV}$.



(14 points)

Solution: Place the cube in a coordinate system with edge 1: $T(0, 0, 0)$, $U(1, 0, 0)$, $V(1, 1, 0)$, $W(0, 1, 0)$, and P, Q, R, S above them, so $P(0, 0, 1)$. Then $A = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $B = \left(1, \frac{1}{2}, \frac{1}{2}\right)$. A normal vector of the plane PAB is

$$\overrightarrow{PA} \times \overrightarrow{PB} = \left(\frac{1}{2}, \frac{1}{2}, -1\right) \times \left(1, \frac{1}{2}, -\frac{1}{2}\right) = \left(\frac{1}{4}, -\frac{3}{4}, -\frac{1}{4}\right)$$

so the plane is $x - 3y - z + 1 = 0$. The points of UV are $(1, t, 0)$, and $1 - 3t + 1 = 0$ gives $t = \frac{2}{3}$. So $UX = \frac{2}{3}$, $XV = \frac{1}{3}$, and $\frac{UX}{XV} = 2$.

8. The real numbers x, y, z satisfy

$$x + y = z - 1, \quad xy = z^2 - 7z + 10$$

For which values of x, y, z is the sum $x^2 + y^2$ maximal?

(15 points)

Solution: We express $x^2 + y^2$ in terms of z :

$$x^2 + y^2 = (x + y)^2 - 2xy = (z - 1)^2 - 2(z^2 - 7z + 10) = -z^2 + 12z - 19$$

But z cannot be arbitrary: x and y are real roots of $t^2 - (z - 1)t + (z^2 - 7z + 10) = 0$, so the discriminant is non-negative:

$$(z - 1)^2 - 4(z^2 - 7z + 10) \geq 0, \quad 3z^2 - 26z + 39 \leq 0, \quad \frac{13 - 2\sqrt{13}}{3} \leq z \leq \frac{13 + 2\sqrt{13}}{3}$$

that is, approximately $1.93 \leq z \leq 6.74$. The function $-z^2 + 12z - 19$ has its maximum at $z = 6$, which is in this interval, and the maximum is 17. Then $x + y = 5$ and $xy = 4$, so the maximum $x^2 + y^2 = 17$ is attained at $x = 1, y = 4, z = 6$ and at $x = 4, y = 1, z = 6$.

Economics Entrance Exam - 1987

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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. Find the exact value of the given expression without using a calculator or tables.

$$\frac{5 \cdot 8^{-\frac{2}{3}} + \frac{2}{\sqrt{3}} \left(\sin \frac{16\pi}{3} \right)^3}{\log_4 2}$$

(9 points)

Solution: $8^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = \frac{1}{4}$. Since $\frac{16\pi}{3} = 4\pi + \frac{4\pi}{3}$, $\sin \frac{16\pi}{3} = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, and its cube is $-\frac{3\sqrt{3}}{8}$. Also $\log_4 2 = \frac{1}{2}$. So the value is

$$\frac{\frac{5}{4} + \frac{2}{\sqrt{3}} \cdot \left(-\frac{3\sqrt{3}}{8} \right)}{\frac{1}{2}} = \frac{\frac{5}{4} - \frac{3}{4}}{\frac{1}{2}} = 1$$

2. The vertices A and B of parallelogram $ABCD$ are $A(1, 4)$ and $B(6, 6)$. The point $P(10, 18)$ is on the line BC , and the point $R(-1, 11)$ is on the line CD . Find the perimeter of the parallelogram. (11 points)

Solution: The line BC goes through $B(6, 6)$ and $P(10, 18)$, with direction $(4, 12)$, which is parallel to $(1, 3)$. So its points are $(6 + t, 6 + 3t)$. The line CD is parallel to AB , whose direction is $(5, 2)$, and goes through R : its points are $(-1 + 5s, 11 + 2s)$. At C :

$$6 + t = -1 + 5s, \quad 6 + 3t = 11 + 2s$$

so $t = 5s - 7$, $13s = 26$, $s = 2$, $t = 3$, and $C = (9, 15)$. So $AB = \sqrt{29}$ and $BC = \sqrt{9 + 81} = 3\sqrt{10}$, and the perimeter is

$$2(\sqrt{29} + 3\sqrt{10}) \approx 29.74$$

3. The base of an isosceles triangle is 36 units long, and its legs are 30 units long. How far is the centroid of the triangle from the center of its circumscribed circle? (12 points)

Solution: Place the base on the x -axis: $(-18, 0)$, $(18, 0)$, and the apex at $(0, h)$ with $h = \sqrt{30^2 - 18^2} = 24$. The centroid is $(0, 8)$. The center of the circumscribed circle is $(0, k)$ with

$$18^2 + k^2 = (24 - k)^2, \quad 324 = 576 - 48k, \quad k = \frac{21}{4}$$

So the distance is $8 - \frac{21}{4} = \frac{11}{4}$.

4. Find the largest subset of the real numbers on which the following expressions are defined.

a) $\frac{\sqrt{x^2 + 4x - 5}}{|x + 2| - 3}$ (8 points)

b) $\log(1 - \sin 2x)$ (5 points)

Solution: a) We need $x^2 + 4x - 5 = (x + 5)(x - 1) \geq 0$, that is, $x \leq -5$ or $x \geq 1$, and $|x + 2| \neq 3$, that is, $x \neq 1$ and $x \neq -5$. So the expression is defined for $x < -5$ and for $x > 1$.

b) We need $1 - \sin 2x > 0$, that is, $\sin 2x \neq 1$, so $2x \neq \frac{\pi}{2} + 2k\pi$. The expression is defined for every real x except $x = \frac{\pi}{4} + k\pi$.

5. The base of a triangular pyramid is an equilateral triangle, and its lateral faces are congruent isosceles triangles. The cross-section through one lateral edge, perpendicular to the opposite base edge, has an area of 150 cm^2 . The volume of the pyramid is 1500 cm^3 . What angle does each lateral face make with the base? (13 points)

Solution: Let the base be the equilateral triangle ABC with side a , the apex D , and the height m . The apex is above the center O of the base. The cross-section through DA perpendicular to BC is the triangle ADF , where F is the midpoint of BC ; its base is $AF = \frac{\sqrt{3}}{2}a$ and its height is m . So

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} am = 150, \quad V = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} a^2 m = 1500$$

Dividing the second equation by the first one: $\frac{a}{3} = 10$, so $a = 30$, and then $m = \frac{600}{\sqrt{3} \cdot 30} = \frac{20\sqrt{3}}{3}$.

The angle φ between the face DBC and the base is the angle DFO , where $OF = \frac{\sqrt{3}}{6}a = 5\sqrt{3}$. So

$$\tan \varphi = \frac{m}{OF} = \frac{20\sqrt{3}/3}{5\sqrt{3}} = \frac{4}{3}, \quad \varphi \approx 53.13^\circ$$

Recommended for students from academic high schools

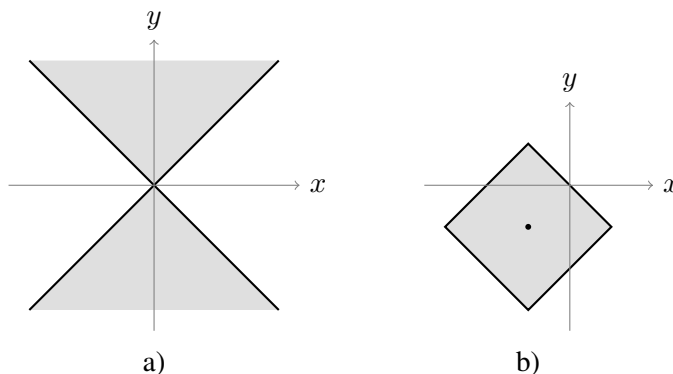
6. Sketch the set of points $P(x, y)$ in the coordinate plane whose coordinates satisfy the following conditions.

a) $|x| \leq |y|$ (5 points)

b) $|x + 1| + |y + 1| \leq 2$ (8 points)

Solution: a) The condition $|x| \leq |y|$ describes the points that are at most as far from the y -axis as from the x -axis (since $|x|$ is the distance from the y -axis and $|y|$ is the distance from the x -axis): the region between the lines $y = x$ and $y = -x$ that contains the y -axis, including the two lines.

b) The condition says that the sum of the distances of (x, y) from the lines $x = -1$ and $y = -1$ is at most 2. This is the square (with its interior) whose vertices are $(1, -1)$, $(-1, 1)$, $(-3, -1)$, $(-1, -3)$, centered at $(-1, -1)$.



7. Solve the given system of equations over the real numbers.

$$x^{\log_8 y} + y^{\log_8 x} = 4$$

$$\log_4 x - \log_4 y = 1$$

(14 points)

Solution: We need $x > 0$ and $y > 0$. Notice that $x^{\log_8 y} = 8^{\log_8 x \cdot \log_8 y} = y^{\log_8 x}$, so the first equation says $2x^{\log_8 y} = 4$, that is, $x^{\log_8 y} = 2$. Let $u = \log_8 x$ and $v = \log_8 y$. Then the first equation is $8^{uv} = 2$, so $uv = \frac{1}{3}$. The second equation says $\frac{x}{y} = 4$, so $u - v = \log_8 4 = \frac{2}{3}$. Then

$$u \left(u - \frac{2}{3} \right) = \frac{1}{3}, \quad 3u^2 - 2u - 1 = 0, \quad u = 1 \text{ or } u = -\frac{1}{3}$$

If $u = 1$: $v = \frac{1}{3}$, so $x = 8$, $y = 2$. If $u = -\frac{1}{3}$: $v = -1$, so $x = \frac{1}{2}$, $y = \frac{1}{8}$. (Check: $8^{\log_8 2} + 2^{\log_8 8} = 2 + 2 = 4$, and $\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{8}\right)^{-\frac{1}{3}} = 2 + 2 = 4$.)

8. Let O be a point in the plane, and let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be unit vectors such that O is outside the triangle ABC . Prove that

$$|\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}| > 1$$

(15 points)

Solution: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the three unit vectors. Since O is outside the triangle ABC , there is a line through O such that the whole triangle is on one side of it. So the points A , B , C of the unit circle around O are on an open half circle. Let B be the middle one of the three points on this half circle, and let $\alpha = \angle AOB$ and $\gamma = \angle BOC$; then $\alpha + \gamma < 180^\circ$. Now

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{b} = \cos \alpha + 1 + \cos \gamma$$

Since $\alpha < 180^\circ - \gamma$, $\cos \alpha > \cos(180^\circ - \gamma) = -\cos \gamma$, so $\cos \alpha + \cos \gamma > 0$. So

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}| \geq (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{b} > 1$$

(the length of a vector is at least its dot product with a unit vector).

Recommended for students from vocational secondary schools

6. Solve the given inequality over the real numbers.

$$2 \log_x \frac{1}{4} + \log_4 \frac{1}{x} \leq -3$$

(13 points)

Solution: We need $x > 0$ and $x \neq 1$. Let $t = \log_4 x \neq 0$. Then $\log_x \frac{1}{4} = -\frac{1}{t}$ and $\log_4 \frac{1}{x} = -t$, so

$$-\frac{2}{t} - t \leq -3, \quad t + \frac{2}{t} - 3 \geq 0, \quad \frac{(t-1)(t-2)}{t} \geq 0$$

A sign chart gives $0 < t \leq 1$ or $t \geq 2$, that is, $1 < x \leq 4$ or $x \geq 16$.

7. In triangle ABC , the median AA_1 is perpendicular to AB . Let a , b , c be the lengths of the sides BC , CA , AB . Express the cosine of the angle CAB in terms of b and c , and the cosine of the angle BCA in terms of a and b . (14 points)

Solution: Use vectors from A : $\overrightarrow{AA_1} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}$. Since $AA_1 \perp AB$,

$$(\overrightarrow{AB} + \overrightarrow{AC}) \cdot \overrightarrow{AB} = 0, \quad c^2 + bc \cos \angle CAB = 0, \quad \cos \angle CAB = -\frac{c}{b}$$

By the Law of Cosines, $a^2 = b^2 + c^2 - 2bc \cos \angle CAB = b^2 + 3c^2$, so $c^2 = \frac{a^2 - b^2}{3}$. Then

$$\cos \angle BCA = \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + b^2 - \frac{a^2 - b^2}{3}}{2ab} = \frac{a^2 + 2b^2}{3ab}$$

8. Let a and b be real numbers with $a > b$ and $ab = 1$. Prove that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2}$$

(15 points)

Solution: Since $ab = 1$, $a^2 + b^2 = (a - b)^2 + 2ab = (a - b)^2 + 2$, so with $d = a - b > 0$,

$$\frac{a^2 + b^2}{a - b} = d + \frac{2}{d} \geq 2\sqrt{d \cdot \frac{2}{d}} = 2\sqrt{2}$$

by the inequality between the arithmetic and geometric means.